(1) Decide whether each of the following series converge absolutely, converge conditionally, or diverge. Explain why you made your decision and show your work.

(a) $\sum_{k=0}^{\infty}(-1)^k \frac{2^{2k+1}}{(k!)^2}$ $\quad a_k = \frac{(-1)^k 2^{2k+1}}{(k!)^2} \quad a_{k+1} = \frac{(-1)^{k+1} 2^{2k+3}}{[(k+1)!]^2}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\cancel{2^{2k}} \cdot 8 \quad \cancel{(k!)^2}}{(k+1)^2 \cancel{(k!)^2}\, \cancel{2^{2k}} \cdot 2} = \frac{4}{(k+1)^2}$$

$$\lim_{k\to+\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to+\infty} \frac{4}{(k+1)^2} = 0 < 1$$

∴ Series converges absolutely.

(b) $\sum_{k=2}^{\infty}(-1)^k \frac{k^2}{2(k^2-1)}$ $\quad$ Alternating Series $\sum_{k=2}^{\infty}(-1)^k a_k$

$$\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} \frac{k^2}{2(k^2-1)} = \frac{1}{2} \neq 0$$

By ~~divergence test~~, Alternating series test, ~~the~~ series diverges.

Divergence Test yields this result as well

(c) $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k^2+4}}$

Compare with series $\sum_{k=2}^{+\infty} \frac{1}{k}$.

Comp Test: $\underbrace{\frac{1}{\sqrt{k^2+4}}}_{b_k} < \frac{1}{\sqrt{k^2}} = \underbrace{\frac{1}{k}}_{a_k}$ is inconclusive.

Limit Comparison Test: $\lim_{k\to+\infty}\left(\frac{a_k}{b_k}\right) = \lim_{k\to+\infty}\left(\frac{\sqrt{k^2+4}}{k}\right) = 1 \neq 0$, finite.

So since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges,

so does $\sum_{k=2}^{+\infty} \frac{1}{\sqrt{k^2+4}}$ DIVERGES

(2) (a) Write the repeating decimal expression .023023023···· as a rational number (ratio of integers) by expressing it as a geometric series and computing the sum of that series.

$$\frac{23}{1000} + \frac{23}{1000000} + \cdots \qquad a = \frac{23}{1000} \qquad r = \frac{1}{1000} < 1$$

Converges to Sum $\dfrac{a}{1-r} = \dfrac{\frac{23}{1000}}{\frac{999}{1000}} = \boxed{\dfrac{23}{999}}$

(b) Show whether or not the integral $\int_2^\infty \frac{x^2-3}{2x^4+7x-2}dx$ converges to a finite number by comparison with a series.

Integral test says that series $\sum_{k=2}^{+\infty} \left(\frac{k^2-3}{2k^4+7k-2}\right)$ converges if and only if the given integral converges.

By limit comparison test, this series may be compared to convergent p-series $\sum_{k=2}^{+\infty} \frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{(k^2-3)\,k^2}{(2k^4+7k-2)(1)} = \frac{1}{2} \neq 0, \text{ finite.}$$

∴ series converges ⟹ $\boxed{\text{Integral converges}}$

(3a) Find the interval of convergence for the following power series: $\sum_{k=1}^{\infty}(-1)^{2k}\frac{(2x-5)^{2k}}{k}$

$$a_K = (-1)^{2K}\frac{(2x-5)^{2K}}{K} = \frac{(2x-5)^{2K}}{K}$$

$$a_{K+1} = \frac{(2x-5)^{2K+2}}{K+1}$$

$$\lim_{K\to+\infty}\left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to+\infty}\frac{|2x-5|^{2K+2}\,K}{(K+1)\,|2x-5|^{2K}} = |2x-5|^2 < 1 \quad \text{for Abs. conv.}$$

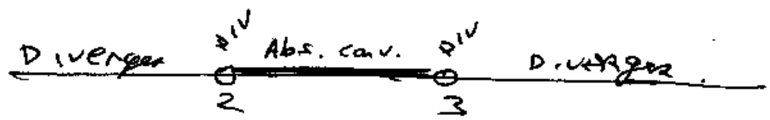
$$\Rightarrow |2x-5| < 1$$

$$\Rightarrow -1 < 2x-5 < +1$$

$$4 < 2x < 6$$

$$\boxed{2 < x < 3}$$

$x = 2 \Rightarrow \sum_{K=1}^{\infty}\frac{(-1)^{2K}}{K} = \sum_{1}^{\infty}\frac{1}{K}$ Diverges



$x = 3 \Rightarrow \sum_{1}^{\infty}\frac{1}{K}$ diverges

(3b) Use the identity $\sin^2 x = ½(1 - \cos 2x)$ and a known Maclaurin series to derive the Maclaurin series expansion for $\sin^2 x$. Express your answer in sigma (Σ) notation.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{K=0}^{\infty}(-1)^K\frac{x^{2K}}{(2K)!}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots = \sum_{K=0}^{\infty}(-1)^K\frac{(2x)^{2K}}{(2K)!}$$

$$= \sum_{K=0}^{\infty}(-1)^K\frac{2^{2K}x^{2K}}{(2K)!}$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left[1 - \sum_{K=0}^{\infty}(-1)^K\frac{4^K x^{2K}}{(2K)!}\right]$$

$$= \frac{1}{2} + \sum_{K=0}^{\infty}(-1)^{K+1}\frac{4^K x^{2K}}{2\cdot(2K)!}$$

(or several variants of this.)

(4a) What approximate value do you get for ln(1.05) by using up to the third order terms of the Maclaurin series for $\ln(1 + x)$?

$f(x) = \ln(1+x)$

$f'(x) = \dfrac{1}{1+x}$

$f''(x) = \dfrac{-1}{(1+x)^2}$

$f'''(x) = \dfrac{2}{(1+x)^3}$

$$\ln(1+x) \cong x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots$$

using terms of to 3rd order

$$\ln(1.05) \cong .05 - \tfrac{1}{2}(.05)^2 + \tfrac{1}{3}(.05)^3$$

$$= .05 - \frac{.0025}{2} + \frac{.000125}{3}$$

$$= .05 - .00125 + .00004167$$

$$\begin{array}{r} .05004167 \\ .00125000 \\ \hline \boxed{.04879167} \end{array}$$

(4b) Give an upper bound for the error in this approximation, using the estimate from Taylor's Theorem.

$$\text{Error} \leq \frac{|f^{(4)}(c)|}{4!}(.05)^4 \quad \text{for some } c \text{ between } 0 \text{ and } .05.$$

$$< \frac{6}{24} \cdot .00000625 = \underline{.0000015625}$$

$f''''(x) = \dfrac{-6}{(1+x)^4}$

$|f^{(4)}(x)| \leq 6$ on this interval.

(4c) What is the least number of terms of this series that would be necessary in order to guarantee accuracy within an error of 0.001?

For this, we could stop at $\ln(1+x) \cong x - \tfrac{1}{2}x^2$

$\boxed{n = 2}$. (using Alternating Series Remainder estimate.)

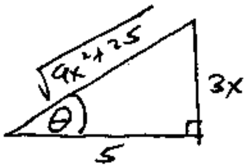
using Taylor estimate, $|R_2(.05)| \leq \dfrac{2}{3!}(.05)^3$

$$= \tfrac{1}{3}(.000125)$$

$$= .0004167 < .001$$

(5) Find the indicated integrals:

(a) $\int \frac{4}{\sqrt{9x^2+25}}\,dx = \int \frac{4 \cdot \frac{5}{3}\sec^2\theta\, d\theta}{5\sec\theta}$



$= \frac{4}{3}\int \sec\theta\, d\theta$

$= \frac{4}{3}\ln|\sec\theta + \tan\theta| + C'$

$\tan\theta = \frac{3x}{5}$

$x = \frac{5}{3}\tan\theta$

$dx = \frac{5}{3}\sec^2\theta\, d\theta$

$\sqrt{9x^2+25} = 5\sec\theta$

$= \frac{4}{3}\ln\left|\frac{\sqrt{9x^2+25}}{5} + \frac{3x}{5}\right| + C'$

$= \boxed{\frac{4}{3}\ln\left|3x + \sqrt{9x^2+25}\right| + C}$

(b) $\int_3^{+\infty} \frac{dx}{(5x+1)^{3/2}} = \frac{1}{5}\int_{16}^{+\infty} \frac{du}{u^{3/2}} = \frac{1}{5}\lim_{T\to+\infty}\int_{16}^{T} u^{-3/2}\,du$

$u = 5x+1$

$du = 5\,dx$

$dx = \frac{1}{5}du$

$= \frac{1}{5}\lim_{T\to+\infty}\left[\frac{2u^{-1/2}}{-1}\right]\Bigg|_{16}^{T}$

$= \frac{2}{5}\lim_{T\to+\infty}\left[\frac{-1}{\sqrt{T}} + \frac{1}{4}\right] = \frac{2}{5}\cdot\frac{1}{4} = \boxed{\frac{1}{10}}$

(c) $\int x^2\cos 5x\,dx = \frac{1}{5}x^2\sin 5x - \frac{2}{5}\int x\sin 5x\,dx$

$u = x^2 \quad dv = \cos 5x\,dx$

$du = 2x\,dx \quad v = \frac{\sin 5x}{5}$

$u = x \quad dv = \sin 5x\,dx$

$du = dx \quad v = -\frac{\cos 5x}{5}$

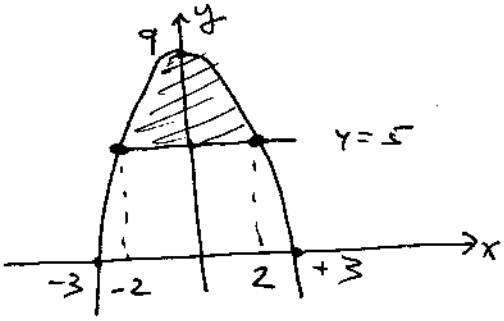
$\Rightarrow -\frac{1}{5}x\cos 5x + \frac{1}{5}\int\cos 5x\,dx$

$= -\frac{1}{5}x\cos 5x + \frac{1}{25}\sin 5x.$

$\Rightarrow \frac{1}{5}x^2\sin 5x - \frac{2}{5}\left[-\frac{1}{5}\cos 5x + \frac{1}{25}\sin 5x\right] + C$

$= \boxed{\left(\frac{1}{5}x^2 - \frac{2}{125}\right)\sin 5x + \frac{2}{25}\cos 5x + C}$

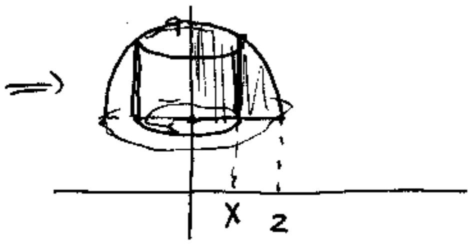
(6) Find the volume of the solid of revolution obtained by revolving about the y-axis the region bounded between the line $y = 5$ and the parabola $y = 9 - x^2$.



MEETS where $9 - x^2 = 5$

$$x^2 = 4$$

$$x = \pm 2.$$



By cylindrical shells,

$$A = 2\pi r h$$

$$= 2\pi x(9 - x^2 - 5)$$

$$= 2\pi x(4 - x^2) = A(x)$$

$$V = \int_0^2 A(x)\,dx$$

$$= 2\pi \int_0^2 (4x - x^3)\,dx$$

$$= 2\pi \left[2x^2 - \frac{x^4}{4}\right]\Big|_0^2 = 2\pi[8 - 4] = \boxed{8\pi}$$

(7) Consider the function:

$$f(x) = (1+2x)^{3/x}$$

a) Find $\lim_{x \to 0^+} f(x)$

$$y = (1+2x)^{3/x}$$

$$\ln y = \frac{3}{x} \ln(1+2x)$$

$$\lim_{x \to 0^+} \frac{3 \ln(1+2x)}{x} \to 0$$

by L'Hôpital $\therefore = \lim_{x \to 0^+} \dfrac{3\left(\frac{2}{1+2x}\right)}{1} = 6$

$$\therefore \ln y \to 6$$

$$y \to \boxed{e^6}$$

b) Find $f'(1)$

$$y = (1+2x)^{3/x}$$

$$\ln y = \frac{3 \ln(1+2x)}{x}$$

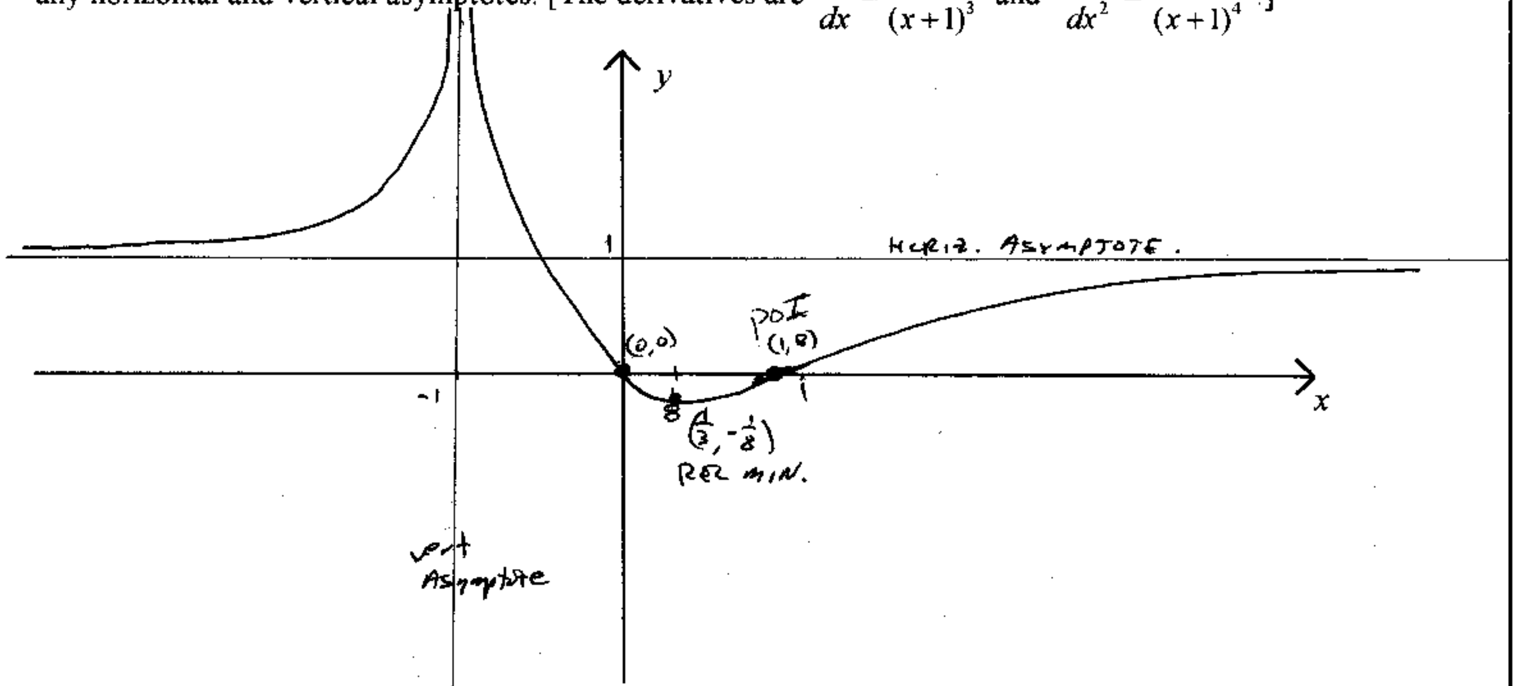
$$\frac{1}{y}\frac{dy}{dx} = \frac{x \cdot 3\left(\frac{2}{1+2x}\right) - 3\ln(1+2x)}{x^2} = \frac{\left(\frac{6x}{1+2x}\right) - 3\ln(1+2x)}{x^2}$$

$$\frac{dy}{dx} = \left[\frac{\left(\frac{6x}{1+2x}\right) - 3\ln(1+2x)}{x^2}\right](1+2x)^{3/x} = f'(x)$$

$$\text{So } f'(1) = \left[\frac{\frac{6}{3} - 3\ln(3)}{1}\right] 3^3 = \left(\frac{2 - 3\ln 3}{1}\right) \cdot 27$$

$$= \boxed{27(2 - 3\ln 3)}$$

(8) Sketch the graph of $y = \frac{x^2 - x}{(x+1)^2}$. Plot any intercepts, stationary points, and points of inflection. Show any horizontal and vertical asymptotes. [The derivatives are $\frac{dy}{dx} = \frac{3x-1}{(x+1)^3}$ and $\frac{d^2y}{dx^2} = \frac{-6x+6}{(x+1)^4}$.]



$f(x) = \frac{x(x-1)}{(x+1)^2} \Rightarrow$ x-int. at $x=0, x=1$.

vert. Asymptote at $x = -1$.

$f'(x) = \frac{3x-1}{(x+1)^3} \Rightarrow$ CRIT PT. at $x = \frac{1}{3}$

$f''(\frac{1}{3}) = \frac{6(\frac{2}{3})}{(\frac{4}{3})^4} > 0 \Rightarrow$ REL MIN

$f''(x) = \frac{6(1-x)}{(x+1)^4} \Rightarrow$ ~~POSSIBLE~~ POI at $x = 1$

$\lim_{x \to +\infty} \frac{x^2 - x}{(x+1)^2} = 1$

$\Rightarrow$ $\boxed{y = 1 \text{ A HORIZ. ASYMPTOTE}}$

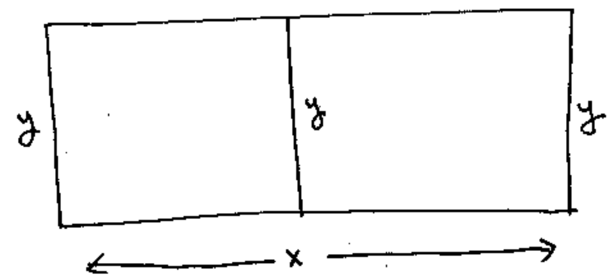
VALUES

x	f(x)	f'(x)	f''(x)	
$x < -1$	+	+	+	
$x = -1$	VERTICAL ASYMPTOTE			
$-1 < x < 0$	—	—	+	
$x = 0$	0	-1	6	(x-int.)
$0 < x < \frac{1}{3}$	—	—	+	
$x = \frac{1}{3}$	$-\frac{1}{8}$	0	+	(REL MIN)
$\frac{1}{3} < x < 1$	—	+	+	
$x = 1$	0	$1/4$	0	(POI), (x-int.)

$f(\frac{1}{3}) = \frac{\frac{1}{3}(-\frac{2}{3})}{(\frac{4}{3})^2} = \frac{-2}{16} = -\frac{1}{8}$

$f'(1) = \frac{2}{8} = \frac{1}{4}$

(9) A farmer wishes to fence a rectangular pasture having a total area of 12,000 square meters. She wants to divide it into two parts with a fence across the middle. Fencing around the outside costs \$7.50 per meter, but the farmer can use less expensive fencing at \$3 per meter as the divider. What dimensions will result in the least cost?



$$C = (2x+2y)(7.5) + (y)(3) = 15x + 18y$$

Constraint: $xy = 12000$ $\qquad y = \frac{12000}{x}$

So $C(x) = 15x + 18\left(\frac{12000}{x}\right) \qquad x > 0.$

$C'(x) = 15 - \frac{18 \cdot 12000}{x^2} = 0$ for crit pt.

$$\frac{18 \cdot 12000}{x^2} = 15$$

$$18 \cdot 12000 = 15x^2$$

$$(\cancel{3} \cdot 6)(1200)(\cancel{10}^{2}) = \cancel{5} \cdot \cancel{3} \cdot x^2$$

$$14400 = x^2$$

$$x = 120 \text{ meters.}$$

$$y = \frac{12000}{x} = \frac{12000}{120} = 100 \text{ meters.}$$

Check: $C''(x) = \frac{36 \cdot 12000}{x^3} > 0 \Rightarrow$ MINIMUM

(10) A rapid transit trolley moves with a constant acceleration and covers the distance between two points 300 feet apart in 8 seconds. Its velocity as it passes the second point is 50 feet per second.

(a) What is its acceleration? $a =$ $\frac{25}{8} = 3.125 \text{ ft/sec}^2$

(b) What is the velocity of the trolley as it passes the first point? $v_0 =$ 25 ft/sec

$t=0$ | accel $= a$ | $t=8$

$v = v_0$ | | $v = 50 \text{ ft/sec}$

$x = 0$ | | $x = 300 \text{ ft}.$

$$\frac{dv}{dt} = a$$

$$v = at + C$$

$$v(0) = C = v_0 \Rightarrow \boxed{v(t) = \cancel{\ } \; v_0 + at}$$

$$\frac{dx}{dt} = v_0 + at$$

$$x(t) = C_2 + v_0 t + \tfrac{1}{2} a t^2$$

$$x(0) = C_2 = 0 \Rightarrow \boxed{x(t) = v_0 t + \tfrac{1}{2} a t^2}$$

$$v(8) = v_0 + 8a = 50$$

$$x(8) = 8v_0 + 32a = 300$$

solve simultaneously:

$$8v_0 + 64a = 400$$

$$8v_0 + 32a = 300$$

$$32a = 100$$

$$\boxed{a = \frac{100}{32} = \frac{25}{8} \text{ ft/sec}^2}$$

$$v_0 = 50 - 8a = 50 - 25 = \boxed{25 \text{ ft/sec} = v_0}$$