Chapter 12

Logarithmic Functions

12.1 Introductory Example

Example 1. A large lake has been serving as a reservoir for its nearby towns. Over the years industries on the shores of the lake have contributed to the pollution of the lake. An awareness of the problem has caused community members to ban further pollution. Due to a combination of runoff and of natural processes, the amount of pollutants in the lake is expected to decrease at a rate proportional to pollution levels. The number of grams of pollutant in the lake is now 1200; t years from now the number of grams is expected to be given by $1200(10)^{-t/8}$. If the water is deemed safe to drink when the pollutant level has dropped to 400 grams, for how many years will the towns need to find an alternate source of drinking water?

We must find t such that $400 = 1200(10)^{-t/8}$. This is equivalent to solving $\frac{400}{1200} = (10)^{-t/8}$, or $\frac{1}{3} = (10)^{-t/8}$.

We can approximate the solution by using a graphing calculator. One approach is to look for the root of $\frac{1}{3} - (10)^{-t/8}$. Another is to look for the point of intersection of $y = 1200(10)^{-t/8}$ and $y = 400$.

But suppose we would like an exact answer; we want to solve the equation analytically for t.
We could simplify somewhat by converting $\frac{1}{3} = (10)^{-t/8}$ to $\frac{1}{3} = (10^t)^{-1/8}$ and raising both sides of this equation to the (-8) to get $\left(\frac{1}{3}\right)^{-8} = 10^t$. We know $\left(\frac{1}{3}\right)^{-8} = \left((3)^{-1}\right)^{-8} = (3)^8 = 6561$, so we must solve the equation

$$10^t = 6561.$$

t is the number we must raise 10 to in order to get 6561. Since $10^3 = 1,000$ and $10^4 = 10,000$, can be sure that t is a number between 3 and 4. Again we could revert to our calculator to get better and better estimates of the value of t. However, if we can find the inverse of the function 10^t then we can find the exact solution to the equation $10^t = 6561$. If $f(t) = 10^t$, then $t = f^{-1}(6561)$.

The Inverse of $\mathbf{f(x) = 10^x}$

We know the function $f(x) = 10^x$ is invertible because it is 1-to-1. The inverse function, f^{-1}, is obtained by interchanging the input and output of f; the graph of f^{-1} can be drawn by reflecting the graph of

10^x over the line $y = x$.

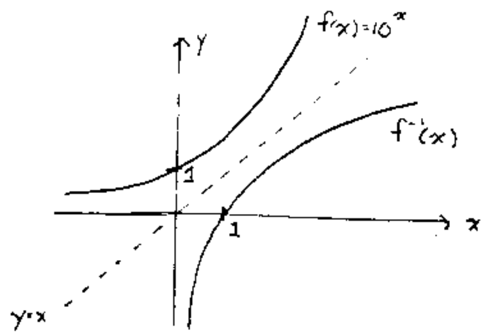

Figure 1.

Suppose we go about looking for a formula for f^{-1} in the usual way, by interchanging the the roles of x and y and solving for y. We write $x = 10^y$. What is y? y is the number we must raise 10 to in order to get x. We don't have an algebraic formula for this, but this function is quite useful, so we give it a name.

Definition. $\log_{10} x$ is the number we must raise 10 to in order to obtain x. $\log_{10} x$ is often written $\log x$. We read $\log_{10} x$ as "log base 10 of x."

$$\boxed{y = \log_{10} x \text{ is equivalent to } 10^y = x.}$$

The domain of f is $(-\infty, \infty)$ and the range is $(0, \infty)$. Therefore the domain of $\log x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. $\log x$ is defined only for $x > 0$. The graph of $\log x$ is increasing and concave down; it is increasing without bound, but it is increasing *very* slowly.

$$\lim_{x \to 0^+} f^{-1}(x) = -\infty, \qquad \lim_{x \to \infty} f^{-1}(x) = +\infty.$$

Note that although we have a nice, compact way of expressing the inverse function of 10^x, it might seem that all we have really accomplished so far is the introduction of a shorthand for writing "the number we must raise 10 to in order to obtain x". But there is a definite perk. A calculator will give a numerical estimate of the logarithm up to 10 digits with just a couple of key strokes.[1] In actual fact, many logarithms you'll work with are irrational, so your calculator is giving an approximation of the value — but a nice, quick, accurate approximation it is!

Let's return to solving the equation $10^t = 6561$. Then $t = \log_{10} 6561$. This is the analytic solution of the equation. A calculator tells us that $\log_{10} 6561 \approx 3.817$, so people can start drinking the water from that polluted lake in about 3.817 years from now.[2]

Let's make sure the definition of $\log_{10} x$ is clear by looking at some examples.

Example 2.

a) $\log_{10} 100$ is the exponent to which we must raise 10 in order to get 100. Since $10^2 = 100$, we have $\log_{10} 100 = 2$.

[1] Before there were calculators that could provide this information tables of logarithms were painstakingly constructed and used for reference.

[2] Keep in mind that 3.817 is an approximate solution to $10^t = 6561$. Try it out; $10^{3.817} = 6561.45\ldots$. Even 3.816970038 is only an approximate solution. The exact solution is $\log_{10} 6561$.

b) $\log_{10} 0.1$ is the exponent to which we must raise 10 in order to get 0.1. Since $10^{-1} = 0.1$, we have $\log_{10} = -1$.

c) $\log_{10} 1$ is the exponent to which we must raise 10 in order to get 1. Since $10^0 = 1$, we have $\log_{10} 1 = 0$.

d) $\log_{10} 151$ is the exponent to which we must raise 10 in order to get 151. Since $10^2 = 100$ and $10^3 = 1000$, we know that $\log_{10} 151$ is between 2 and 3. $\log_{10} 151$ is irrational; we can't express it exactly using a decimal.

e) $\log_{10} 0$ is the exponent to which we must raise 10 in order to get 0. There is no such number! $10^{\text{any number}}$ is always positive. Zero is not in the domain of the function $\log_{10} x$ because 0 is not in the range of 10^x, the inverse of $\log_{10} x$. See how your calculator responds when you enter $\log 0$. (It should complain.)

12.2 The Logarithmic Function Defined

Let $f(x) = b^x$ where b is a positive number. Since f is a one-to-one function, it has an inverse function, f^{-1}. From the graph of $f(x) = b^x$ we can obtain the graph of its inverse function. The domain of f^{-1} is $(0, \infty)$ while the range is $(-\infty, \infty)$.

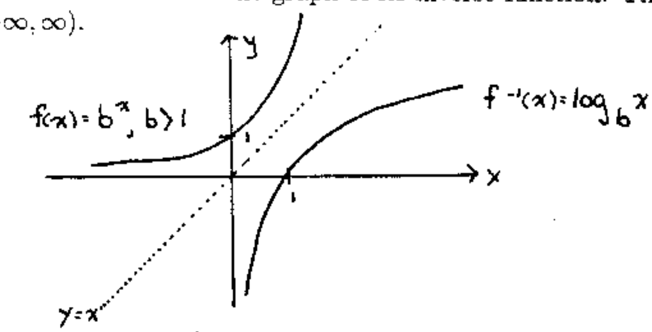

Figure 2.

We look for a formula for the inverse function by interchanging the the roles of x and y and solving for y. We write $x = b^y$. What is y? y is the number we must raise b to in order to get x. As in the case of $f(x) = 10^x$ above, we haven't arrived at an algebraic formula for f^{-1}, but this function whose output is y is so useful that it is given its own name.

Definition. $\log_\mathbf{b} \mathbf{x}$ is the number we must raise b to in order to get x.

We read $\log_b x$ as "log base b of x."

By definition, the following two statements are equivalent.[3]

$\log_b x = y$	$\Leftrightarrow$	$b^y = x$
logarithmic form	is equivalent to	exponential form

[3] $\Leftrightarrow$ is a symbol that means "if and only if" or "is equivalent to". The symbol is a composite of $\Rightarrow$ and $\Leftarrow$ where $A \Rightarrow B$ means A implies B or "if A then B" and $A \Leftarrow B$ means B implies A or "if B then A." In this context $\Rightarrow$ corresponds to the statement: if $y = \log_b x$ then $b^y = x$. $\Leftarrow$ corresponds to the statement if $b^y = x$ then $y = \log_b x$. The composite symbol $A \Leftrightarrow B$ means that the statements A and B carry the same information (in different forms).

Test your understanding of this definition by working through the following exercises.

(Answers to Exercises 1 and 2 are supplied at the end of the chapter.)

Exercise 1. If the statement is given in logarithmic form, write it in exponential form; if it is written in exponential form, write it in logarithmic form.

$$\text{a)}\quad \log_3 81 = 4 \qquad \text{b)}\quad -.5 = \log_2\left(\frac{1}{\sqrt{2}}\right) \qquad \text{c)}\quad A = W^b$$

Exercise 2. Estimate the following logarithms by finding two consecutive integers, one smaller than the logarithm and the other larger.

$$\text{a)}\quad \log_6 40 \qquad \text{b)}\quad \log_{10} 3789 \qquad \text{c)}\quad \log_2 40$$

Exercise 3. We know that $\log_{10} 10 = 1$ and $\log_{10} 100 = 2$. Therefore, $1 < \log_{10} 50.5 < 2$. However, although 50.5 is midway between 10 and 100, $\log_{10} 50.5$ is not midway between 1 and 2, since log is not a linear function. Is $\log_{10} 50.5$ greater than 1.5 or less than 1.5? Argue geometrically. Check your answer using your calculator.

Logarithms and your calculator. A calculator will generally give numerical approximations of logs to two different bases, base 10 and base e.

$\log_{10} x$ is called the **common log of x** and is denoted by **log x.**

$\log_e x$ is called the **natural log of x** and is denoted by **ln x.**[4]

Recall that when we were studying the derivative of $f(x) = b^x$ we found that the derivative of b^x is kb^x where k is the slope of the tangent to b^x at $x = a$. We searched for a base b such that the derivative of b^x is simply b^x. Such a base lies between the values of 2.71 and 2.72. We defined e to be this base. Just as the derivative of e^x is charming in its simplicity, so too will we find charm in the elegant simplicity of the derivative of $\ln x$. It's something to look forward to in the next chapter! (Or try to discover it on your own!) Notice that on your calculator the pair of inverse functions 10^x and $\log x$ share the same key. Similarly, the pair of inverse functions e^x and $\ln x$ share the same key.[5]

Note that

$$y = \ln x \text{ is equivalent to } x = e^y.$$

Since $\ln x$ and e^x are inverse functions,

$$e^{\ln \bigstar} = \bigstar \text{ for all positive } \bigstar, \text{ and } \ln e^{\bigstar} = \bigstar.$$

[4]Yes, the e is invisible in this notation. Soon you will get accustomed to this. In the meantime, if you get confused by the notation $\ln x$ you can write it as $\log_e x$ to clarify the meaning.

[5]Notice too that the inverse functions x^2 and $\sqrt{x}$ share the same key. We will see that the trigonometric functions and their inverse functions also share a key. Your calculator's organization reflects the inverse function relation.

12.3 The Properties of Logarithms

The properties of logarithms can be derived from the properties of exponentials and the inverse relation between logarithms and exponentials.

We know that if f and g are inverse functions then $f(g(x)) = x$ and $g(f((x))) = x$. By definition, the functions $\log_b x$ and b^x are inverse functions. Therefore we have the following inverse function identities

$$\boxed{\log_b b^\bigstar = \bigstar} \quad \text{and} \quad \boxed{b^{\log_b \bigstar} = \bigstar}.$$

where $\bigstar$ is any expression in the former identity, and any positive expression in the latter.

We can also work our way through these by thinking about the meanings of the expressions.

$\log_b x$ is the number we must raise b to in order to get x. Therefore,

$\log_b b^\bigstar$ is the number we must raise b to in order to get $b^\bigstar$. $\log_b b^\bigstar = \bigstar$.

On the other hand,

$b^{\log_b \bigstar}$ is b raised to the number we must raise b to in order to get $\bigstar$. $b^{\log_b \bigstar} = \bigstar$.

If we raise b to the power required to get $\bigstar$ we ought to get $\bigstar$!

Laws of Logarithmic and Exponential Functions

Since we have defined logarithms in terms of exponential functions, we can deduce the laws of working with logarithms from those of exponentials.

Exponent Laws	Logarithm Laws
i) $b^x b^y = b^{x+y}$	i) $\log_b QR = \log_b Q + \log_b R$
ii) $\frac{b^x}{b^y} = b^{x-y}$	ii) $\log_b \left(\frac{Q}{R}\right) = \log_b Q - \log_b R$
iii) $(b^x)^y = b^{xy}$	iii) $\log_b R^p = p \log_b R$

Derivation of Logarithmic Laws:

Let $\log_b R = y$ and $\log_b Q = x$.
Then $b^y = R$ and $b^x = Q$.

i) $\log_b QR = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y = \log_b Q + \log_b R$

ii) $\log_b \left(\frac{Q}{R}\right) = \log_b \left(\frac{b^x}{b^y}\right) = \log_b(b^{x-y}) = x - y = \log_b Q - \log_b R$

iii) $\log_b(R^p) = \log_b((b^y))^p) = \log_b(b^{py}) = py = p \log_b R$

Note:

$\log_b 1 = 0$ since $b^0 = 1$;

$\log_b\left(\frac{1}{R}\right) = \log_b 1 - \log_b R = 0 - \log_b R = -\log_b R$

Caution. Resist the temptation to be footloose, fancy-free, and sloppy with the log identities and laws we've just discussed. These laws mean precisely what they say, not more, not less. A novice might look at the law $\log AB = \log A + \log B$ and incorrectly think 'multiplication and addition are the same for logs.' But this is NOT what the law says.

$$(\log A)(\log B) \neq \log A + \log B.$$

Rather, the law says, that the logarithm of a product is the sum of the logarithms. Similarly, many have succumbed to the temptation to look at

$$b^{\log_b \star} = \bigstar \qquad \text{and draw incorrect conclusions.}$$

$b^{2\log_b k}$ does not equal $2k$, despite the popular appeal. You can't just ignore that 2, use the inverse relation of logs and exponentials and then let the 2 rematerialize, lose altitude and slip in right next to the k. Rather, you must rewrite $b^{2\log_b k}$ so you can apply the identity above.

$$\begin{aligned} b^{2\log_b k} &= b^{\log_b k^2} && \text{by log law (iii)} \\ &= k^2 && \text{by the inverse relation of logs and exponentials} \end{aligned}$$

The examples and exercises that follow will provide practice in applying the log identities and laws. It might be useful to write the log laws out and be sure you can identify precisely which you are using as you work your way through the problems.

Example 3. Simplify the following

$$\text{i) } 7^{[\log_7(x^3)+\log_7 4]} \qquad \text{ii) } 7^{[2\log_7 x+3]}$$

Answers:

$$\text{i) } 7^{[\log_7(x^3)+\log_7 4]} = 7^{\log_7(x^3)} \cdot 7^{\log_7 4} = x^3 4 = 4x^3$$

$$\text{ii) } 7^{[2\log_7 x+3]} = 7^{2\log_7 x} \cdot 7^3 = 7^{\log_7 x^2} \cdot 7^3 = x^2 7^3 = 343x^2$$

Exercise 1. Write the given expression in the form log( ) or ln ( ).

i)	$\log A - 3\log B + \frac{\log C}{2}$	ii)	$\frac{-3\log 7}{5} + \frac{1}{2}\log 49$
iii)	$\ln(x^4 - 4) - \ln(x^2 + 2)$	iv)	$\ln 3x - 3\ln x$

Exercise 2. Simplify

i)	$10^{3\log 2 - 2\log 3}$	ii)	$5^{-.5\log_5 3}$
iii)	$\frac{\log 4 - \log 1}{2}$	iv)	$e^{-\ln\sqrt{x}}$
v)	$\ln\left(\frac{1}{\sqrt{e}}\right)$	vi)	$3\ln e^{\pi}$
vii)	$e^{2\ln x - \ln y}$		

Exercise 3. What is the average rate of change of $\log x$ on the interval [1, 10]? On the interval [10, 100]? How many times bigger is the average rate of change of $\log x$ on [1, 10] than on the interval [10, 100]?

12.4 An Historical Interlude on Logarithmic Functions

The Scotsman John Napier's invention of logarithms provided scientists and mathematicians of the 17th century with a revolutionary computational tool. (Today this is Napier's claim to fame, although in his own time he was also known as a widely published religious activist, a Protestant vehemently opposed to the Catholic Church and its Pope.) At the heart of Napier's logarithmic system was the ingenious idea of exploiting the laws of exponents in order to convert complicated multiplication, division, and exponentiation problems into substantially simpler addition, subtraction, and multiplication problems respectively. Although his system did not exactly reflect the log system we use today, he associated numbers with exponents by creating extensive 'log' tables. It is interesting to note that when Napier first began his work, fractional exponents were not in common use, so he had to choose a small enough base to make his system useful. The number that essentially played the 'base role' was $1 - 10^{-7} = .9999999$.[6] The British mathematician Henry Briggs built upon Napier's work and by 1624 produced accurate log tables using a base of 10.

To add some historical perspective, all of this was going on at about the same time as Johannes Kepler (1571–1630) was painstakingly recording data and doing computations (all by hand, without a calculator!) that led him to conclude that the earth's path around the sun is not a perfect circle, but rather an ellipse. Kepler began his computations without the help of logarithms.

Historically logarithms arose as an invaluable computational tool, a role they have lost with the advent of calculators and computers.

How do we use logarithms today?

In modern day mathematics logarithms have claim to fame in the role of a function. In particular, we will find the logarithm useful in its role as the inverse of the exponential function. For example,

◇ **Logarithms help us solve for variables in exponents**

Look back at the introductory example. Logarithms helped us solve for the variable in the exponent. We'll return to this in the next section.

◇ **Logarithms aid us in modelling and conveying information**

We've seen that many quantities grow exponentially. Try graphing $f(x) = 10^x$ for $0 \leq x \leq 6$. That's not a huge domain; it doesn't seem like an unreasonable request. And yet it's very impractical to convey the information since the vertical scale must reach from 1 to 1,000,000! To

[6]From e The Story of a Number by Eli Maor, Princeton University Press, 1994, pp. 3–9.

graph on scales like this, sometimes a semilog plot is used. A semilog plot has one axis labelled in the usual linear way and the other axis, in order to accommodate the data, labeled with the logarithm of the values. For instance, 1, 10, 100, and 1000 would be spaced at equal intervals.

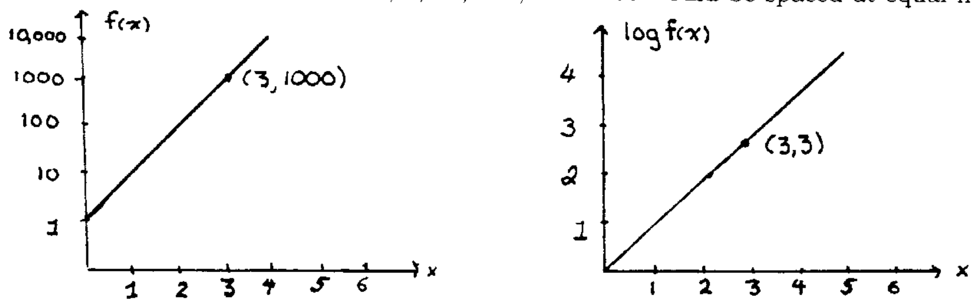

Figure 3.

Many scales of measurements are logarithmic. For example, the Richter scale for earthquakes uses a logarithmic scale. An earthquake measuring 7 on the Richter scale is 10 times stronger than one of magnitude 6, which in turn is 10 times stronger than one of magnitude 5.

Other commonly used logarigthmic scales include decibels for measuring loudness and pH for measuring acidity.

12.5 Using Logarithms and Exponentiation to Solve Equations

The most fundamental ideas for solving equations using logarithms and exponentiation are these:

- If $A = C$ then $b^A = b^C$.

  Exponentiating both sides of an equation preserves equality.

- If $A = C$ where A and C are both positive, then $\log_b A = \log_b C$.

  Taking the logarithm of both sides of an equation preserves equality.

Let's begin with two very simple examples. In both, our goal is to solve for x.

Example 4. Solve for x if $3^{5x} = 100$.

We need to "bring down the exponent" to solve for x, so we need to undo the exponentiation. Logarithms will help us to do this; we'll take the log of both sides of the equation. At first you might think that you *must* use $\log_3$. You *could* do this, but it actually carries a disadvantage. Let's try it and see.

$$\begin{aligned} 3^{5x} &= 100 \\ \log_3(3^{5x}) &= \log_3(100) \\ 5x &= \log_3(100) \\ x &= \frac{\log_3 100}{5} \end{aligned}$$

We've solved for x. But if we want a numerical approximation of our answer, it is not readily available because $\log_3$ is not one of the two log buttons on calculators. We'd have to convert the base 3 logarithm to something our calculators can handle. Let's do this problem again, using $\log x$ (i.e., $\log_{10} x$) and $\ln x$ (i.e., $\log_e x$). You'll see that the truly crucial property of logarithms that allows us to solve the equation is log property (iii), $\log_b(R^p) = p \log_b R$. Therefore we can use $\log_b$ with any base b.

$$\begin{aligned} 3^{5x} &= 100 \\ \log(3^{5x}) &= \log(100) \\ 5x \log 3 &= \log(100) \\ x &= \frac{\log 100}{5 \log 3} = \frac{2}{5 \log 3} \end{aligned} \qquad \begin{aligned} 3^{5x} &= 100 \\ \ln(3^{5x}) &= \ln(100) \\ 5x \ln 3 &= (100) \\ x &= \frac{\ln 100}{5 \ln 3} \end{aligned}$$

Use your calculator to get a numerical approximation of the answer. Do this using both the natural logarithm and the common logarithm. In each case you should come out with the answer of approximately 0.83836. Try it.

[ln] [100] [÷] [(] [5] [ln] [3] [)] [=]

Suppose we had the answer $x = \frac{\log_3 100}{5}$ and wanted to convert to base 10. Is it really necessary to begin the problem again? Of course not! Let's take a look at converting logarithms from one base to another.

Converting logarithms from one base to another:

Goal: We want to convert $\log_b x$ to a logarithm with another base. We begin by writing $y = \log_b x$.

$$y = \log_b x \text{ is equivalent to } b^y = x$$

We can convert $\log_b x$ to a log with *any* base by taking the appropriate logarithm of both sides of the exponential equation $b^y = x$ and solving for y.

Let's convert to the natural logarithm.

$$\begin{aligned} b^y &= x \\ \ln(b^y) &= \ln x \\ y \ln b &= \ln x \\ y &= \frac{\ln x}{\ln b}. \end{aligned}$$

Since b is a constant, $\ln b$ is simply a constant.

Similarly, we could convert from log base b to log base k.

$$\boxed{\log_b x = \frac{\log_k x}{\log_k b}}$$

Returning to our original problem, we convert $\frac{\log_3 100}{5}$ to log base 10 as follows.

$$\frac{\log_3 100}{5} = \frac{1}{5}\frac{\log 100}{\log 3} = \frac{1}{5}\frac{2}{\log 3} = \frac{2}{5\log 3}$$

Example 5. Solve for x if $\log_7(4x) = 3$.

In order to solve for x we need to free the x from the logarithm. There are two approaches we can take to this. The first is to undo $\log_7$ by exponentiating both sides of the equation with base 7.

$$\begin{aligned} \log_7(4x) &= 3 \\ 7^{\log_7 4(x)} &= 7^3 \\ 4x &= 7^3 \\ x &= \frac{7^3}{4} = \frac{343}{4} \end{aligned}$$

Notice that if we use a base other than 7 to exponentiate then we will get stuck. For instance, we could write $10^{\log_7(4x)} = 10^3$, but we cannot simplify the expression $10^{\log_7(4x)}$, so this course of action is unproductive.

An alternate mindset for solving the original problem is to think about what $\log_7(4x) = 3$ means and write the statement in exponential form. $\log_7(4x) = 3$ means "3 is the number we must raise 7 to in order to get $4x$"

So $7^3 = 4x$. Then $x = \frac{7^3}{4} = \frac{343}{4}$.

Theme and Variation: Examples 4 and 5 are prototypical examples. Example 4 is of the form

$$B^{f(x)} = A$$

and Example 5 is of the form

$$\log_b f(x) = A.$$

Knowing our way around equations of these forms will serve as a guideline for strategizing when we have more complicated examples. Below we show variations on these basic themes. Along the way we'll try to point out some pitfalls so you can walk around them instead of falling into them. It's surprisingly easy to either get caught in a frenzy of unproductive manipulation or to plunge down a short dead end street when approaching exponential or logarithmic equations. Take time to strategize.

Example 6. Solve for x if $5^{2x+1} = 20$.

We need to "bring down the exponent" to solve for x. Log property (iii) will help us do this. Again, although we can use $\log_b$ with any base b, if we want a numerical approximation using common logarithms or natural logarithms simplifies the task.

$$
\begin{aligned}
5^{2x+1} &= 20 && \text{Take log of both sides.}^7 \\
\log 5^{2x+1} &= \log 20 && \text{Use log rule (iii); don't forget parentheses around } 2x+1. \\
(2x+1)\log 5 &= \log 20 \\
2x\log 5 + \log 5 &= \log 20 && \text{This is just a linear equation in } x. \text{ (This might not be immediately clear at first, but } \log 5 \text{ is just a constant, as is } \log 20.) \text{ Get all terms with } x \text{ on one side and all else on the other.} \\
2x\log 5 &= \log 20 - \log 5 && \text{Divide by the coefficient of } x. \\
x &= \frac{\log 20 - \log 5}{2\log 5}
\end{aligned}
$$

At this point, you could leave the answer as is, simplify, or look for a numerical approximation. Even if you plan to do the latter, it still might be easier for you to simplify first.

$$\frac{\log 20 - \log 5}{2\log 5} = \frac{\log(20/5)}{\log 25} = \frac{\log 4}{\log 25}.$$ [8]

Example 7. Solve for x if $5^{2x+1} - 18 = 2$.

Caution: We need a strategy here. Our goal is to "bring down the $2x+1$", but taking logs right away doesn't help us. Although $\log(5^{2x+1} - 18) = \log 5$, we can't do much with the expression on the left because there is no log rule to simplify $\log(A+B)$. The x is trapped inside the log with no escape route. However, if we put the equation into the form $B^{f(x)} = A$ by adding 18 to both sides then we're in business. In fact, we're back to $5^{2x+1} = 20$, as in Example 6.

Note:

$$
\begin{aligned}
\text{If } A &= C + D, \quad \text{then} \\
\ln(A) &= \ln(C+D) \quad \text{but} \\
\ln A &\neq \ln C + \ln D.
\end{aligned}
$$

You must take the log of each entire side of the equation.

As a more concrete example,

$$
\begin{aligned}
2 &= 1 + 1 \\
\text{but} \quad \ln 2 &\neq \ln 1 + \ln 1 \\
\ln 2 &\neq 0
\end{aligned}
$$

[7] An alternative strategy is to begin by dividing both sides of the equation by 5 to get $5^{2x} = 4$. Now take logs to get $\log(5^{2x}) = \log 4$, or $2x\log 5 = \log 4$. consequently, $x = \frac{\log 4}{2\log 5}$.

[8] Alternatively $\frac{\log \frac{20}{5}}{2\log 5} = \frac{\log 4}{2\log 5} = \frac{\log 2^2}{2\log 5} = \frac{2\log 2}{2\log 5} = \frac{\log 2}{\log 5}$.

Similarly,

$$\begin{aligned} \text{if } A &= C + D \quad \text{then} \\ e^A &= e^{C+D} = e^C \cdot e^D \quad \text{but} \\ e^A &\neq e^C + e^D \end{aligned}$$

As a more concrete example,

$$\begin{aligned} 2 &= 1 + 1 \\ 10^2 &= 10^{(1+1)} = 100 \quad \text{but} \\ 10^2 &\neq 10^1 + 10^1 = 20 \end{aligned}$$

Example 8. Solve for x if $5^{x+1} - \frac{20}{5^{-x}} = 0$.

Let's strategize. We're trying to solve for a variable in the exponent, so perhaps we can get this into the form $B^{f(x)} = A$. First we'll clean it up a bit; we can clear the denominator by multiplying by 5^x.

$$\begin{aligned} 5^x \left(5^{x+1} - \frac{20}{5^{-x}}\right) &= 5^x(0) \\ 5^{2x+1} - 20 &= 0 \\ 5^{2x+1} &= 20, \text{ and we're back to our familiar problem.} \end{aligned}$$

Example 9. Solve for x. $\log_7(x^3) + \log_7 4 = 2\log_7(x) + 3$

We have plenty of options here. Below we'll work through a couple of approaches.

Approach 1. Try to put this equation into the form $\log_b f(x) = A$.

$\log_7(x^3) + \log_7 4 - 2\log_7(x) = 3$ Consolidate: $\log_7(x^3) + \log_7 4 - 2\log_7(x) = \log_7(x^3) + \log_7 4 - \log_7(x^2) = \log_7\left(\frac{4x^3}{x^2}\right)$

$\log_7\left(\frac{4x^3}{x^2}\right) = 3$ $\frac{4x^3}{x^2} = 4x$; $x = 0$ is not in the domain of $\log_7 x$.

$\log_7(4x) = 3$

Now we're back to Example 4.

Alternatively, we could have grouped terms as follows

$3\log_7(x) - 2\log_7(x) = -\log_7 4 + 3$

$\log_7(x) = -\log_7 4 + 3$ Exponentiating now gives

$7^{\log_7 x} = 7^{-\log_7 4 + 3}$ Careful with the right hand side! $7^{A+B} = 7^A 7^B$, *not* $7^A + 7^B$.

$x = 7^{-\log_7 4} \cdot 7^3$

$x = 7^{\log_7(4^{-1})} \cdot 7^3$

$x = (4^{-1})7^3 = \frac{343}{4}$

Hint: Alternatively $\frac{\log 5}{=}\frac{\log 4}{2\log 5} = \frac{\log 2^2}{2\log 5} = \frac{\log 2}{\log 5}$

Approach 2. Exponentiate immediately. This works, but not as neatly as Approach 1.

$7^{[\log_7(x^3) + \log_7 4]} = 7^{[2\log_7 x + 3]}$

Now use the laws of logs and exponents to simplify.

This is done in Example 3 of §12.3.

$4x^3 = x^2 7^3$ We can divide through by x^2 provided $x \neq 0$.

$4x = 7^3$ or $x = 0$ $x \neq 0$ because 0 is not in the domain of the log function.

$x = \frac{7^3}{4}$ discard the extraneous root $x = 0$

Two Basic Principles:

(1) If you want to solve for a variable and it's caught up in an exponent, you need to bring the variable down. This requires taking the logarithm of both sides of the equation. If you can get the equation in the form

$$B^{f(x)} = A$$

you are in great shape. Taking logs of both sides leaves you with

$$f(x)\log B = \log A$$

$$f(x) = \frac{\log A}{\log B}$$

Since $\frac{\log A}{\log B}$ is independent of x, it is just a constant. Don't be fazed by its bulk; treat it as you would any other constant. You now have $f(x) = k$.

$B^{f(x)} = A^{g(x)}$ is a nice form as well. Taking logs of both sides leaves you with

$$f(x)\log B = g(x)\log A$$

Again, $\log B$ and $\log A$ are just constants.

(2) If you want to solve for a variable and it is caught inside a logarithm, (i.e., in the argument (or input) of the log function), then you'll need to exponentiate to undo the log. Use the same base for exponentiation as is in the log. If you can get your equation in the form

$$\log_b f(x) = C$$

then you can exponentiate both sides of the equation using b as a base to get

$$b^{\log_b f(x)} = b^C$$
$$f(x) = b^C$$

Since b^C is just a constant, you're in good shape.

The Overarching Principle for Solving Equations of the Form g(x) = constant:

Suppose you have an equation of the form $g(x) = k$ where k is an expression without x's. If g is invertible, you can solve for x by undoing g. $g^{-1}(g(x)) = g^{-1}$ (an expression without x's). Logarithms are undone though exponentiation, and exponentials are undone by taking logarithms, since logarithmic and exponential functions are inverses.[9]

Example 10. If we have an equation such as $\frac{1}{[\ln(x+2)]^3} + 7 = 34$, and we want to solve for x, we can think of what was done to x and undo it. Remember that since you put on your socks and then your shoes, your shoes must come off before your socks. To construct the left hand side of the equation we begin with x and do the following:

Add 2; take the natural logarithm; cube the result; take the reciprocal; add 7. To undo this sequence we can subtract 7; take the reciprocal; take the cube root; exponentiate; subtract 2.

Let's do it.	$\frac{1}{[\ln(x+2)]^3} + 7 = 34$	Subtract 7 from both sides.
	$\frac{1}{[\ln(x+2)]^3} = 27$	Take the reciprocal of both sides. (If $A = B$, A and $B \neq 0$, then $1/A = 1/B$.)
	$[\ln(x+2)]^3 = \frac{1}{27}$	Take the cube root of both sides.
	$\ln(x+2) = \frac{1}{3}$	Exponentiate with base e (since ln is $\log_e$).
	$e^{\ln(x+2)} = e^{1/3}$	
	$x + 2 = e^{1/3}$	Subtract 2.
	$x = e^{1/3} - 2$	

Worked examples in solving for x:

(Try these on your own and then read the solutions. Notice that different strategies are adopted depending on the problem.)

Examples:

[9] You may be wondering why when you solve the equation $B^{f(x)} = A$ you can use logs with any base to bring down $f(x)$ while to undo $\log_b$ you must exponentiate with base b. The reason is that $a^x = b^{kx}$ for the appropriate k, where b is any positive number, so a logarithm with any base will do. For example, $5^x = (10^{\log 5})^x = 10^{(\log 5)x} = 10^{kx}$ where k is the constant $\log 5$. Since $5^x = 10^{kx}$, log base 10 will help you out.

11. $3^{2x+1} = 8^x$ — Bring those exponents down; take ln of both sides.
$\ln(3^{2x+1}) = \ln 8^x$
$(2x+1)\ln 3 = x \ln 8$ — Multiply out to free the x.
$2x\ln 3 + \ln 3 = x\ln 8$ — This equation is linear in x.
$2x\ln 3 - x\ln 8 = -\ln 3$
$x(2\ln 3 - \ln 8) = -\ln 3$
$x = \dfrac{-\ln 3}{2\ln 3 - \ln 8}$ or $x = \dfrac{-\ln 3}{\ln(9/8)}$

12. $[\log(x^2+1)]^2 = 4$ — Unpeel the problem as was done above in Example 10.
$\log(x^2+1) = \pm\sqrt{4}$

$\log(x^2+1) = 2$	or	$\log(x^2+1) = -2$
$10^{\log(x^2+1)} = 10^2$		$10^{\log(x^2+1)} = 10^{-2}$
$(x^2+1) = 10^2$		$(x^2+1) = 10^{-2}$
$x^2 = 100 - 1$		$x^2 = .01 - 1 = -.99$
$x = \pm\sqrt{99}$		x^2 can't be negative

13. $\ln\sqrt{x} = \ln x^5 - 7$ — Rewrite this.
$(1/2)\ln x = 5\ln x - 7$ — This equation is linear in $\ln x$.
$(1/2)\ln x - 5\ln x = -7$ — Solve for $\ln x$ and then exponentiate.
$-(9/2)\ln x = -7$
$\ln x = 14/9$
$x = e^{14/9}$

14. $e^{2x} + 3e^x = 10$ — This equation is quadratic in e^x.
$(e^x)^2 + 3e^x - 10 = 0$ — If you like — let $u = e^x$ and solve for u.
Then return to e^x and solve for x.

$u^2 + 3u - 10 = 0$
$(u-2)(u+5) = 0$
$u = 2$ or $u = -5$ — But $u = e^x$.
$e^x = 2$ or $e^x = -5$
$x = \ln 2$ or $e^x = -5$ — But e^x can never be negative, so $e^x \neq -5$.
$x = \ln 2$

15. $5^{\ln x} = 7x$ — Bring down the exponent; take ln of both sides.
$\ln(5^{\ln x}) = \ln(7x)$ — Separate out the $\ln x$'s.
$\ln x(\ln 5) = \ln 7 + \ln x$ — This equation is linear in ln x.
Solve for $\ln x$ and then exponentiate.

$(\ln 5)\ln x - \ln x = \ln 7$
$\ln x(\ln 5 - 1) = \ln 7$
$\ln x = \dfrac{\ln 7}{\ln 5 - 1}$
$x = e^{\left(\frac{\ln 7}{\ln 5 - 1}\right)}$
or $x = \left(e^{\ln 7}\right)^{1/(\ln 5 - 1)} = 7^{1/(\ln 5 - 1)}$

12.6 Graphs of Logarithmic Functions: Theme and Variation

In your mind's eye you should carry a picture of a pair of simple exponential and logarithmic functions, (like 10^x and $\log x$ for instance) because a picture tells you a lot about how the functions behave. "Why clutter my mind? I can always consult my graphing calculator!" you might be thinking. That's a bit like saying "why remember my mother's and my father's names? I can always ask my sister; she knows!" The exponential and logarithmic functions behave so very differently that you want to be able to have identifiers for them. It's not difficult to reconstruct an exponential graph for yourself if necessary, and from that you can construct a logarithmic graph. (If you want to get a feel for e^x, try 3^x.)

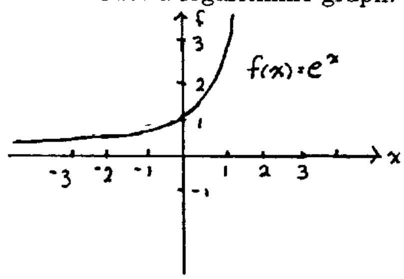

Figure 4.

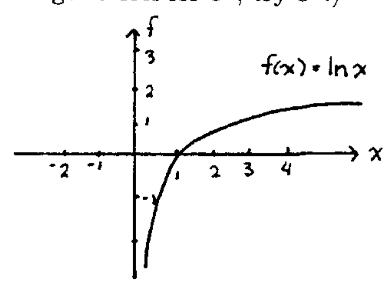

Figure 5.

We briefly recap some important characteristics of the exponential and logarithmic functions.

Exponentials with a base greater than 1 grow increasingly rapidly. A log graph grows increasingly slowly, although the log function grows without bound: as $x \to \infty$, $\log x \to \infty$. Think of $y = \log_{10} x$ for a moment. If $x = 10$ then $y = 1$. To get to a height of 2, x must be 100. To reach a height of 3, x must increase to 1000; for a height of 6, x must be 1 million. This is sluggish growth.

Exponential functions are defined for all real numbers, but the range of b^x is only positive numbers. On the other hand, $\log_b x$ is defined only for positive numbers, while its range is all real numbers. These are not functions you want to confuse.

In this section we will play around a bit with some fancier variations on the basic logarithmic function and its graph.

Example 16. Sketch $f(x) = -\log_2 x$. Label at least three points on the graph.

$\log_2 x$ and 2^x are inverse functions. We are familiar with the graph of 2^x, so we'll begin with the graph of $y = 2^x$ and reflect it over the line $y = x$ (interchanging the x and y coordinates of the points) to obtain the graph of $y = \log_2 x$.

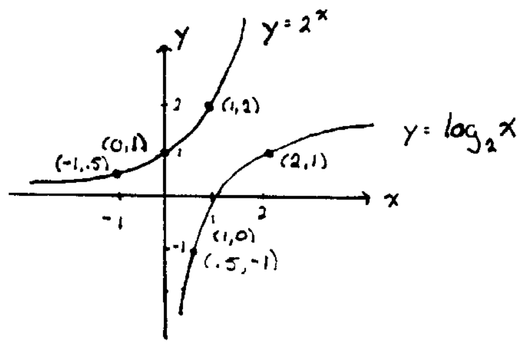

Figure 6.

To obtain the graph of $f(x) = -\log_2 x$ we flip the graph of $\log_2 x$ over the x-axis.

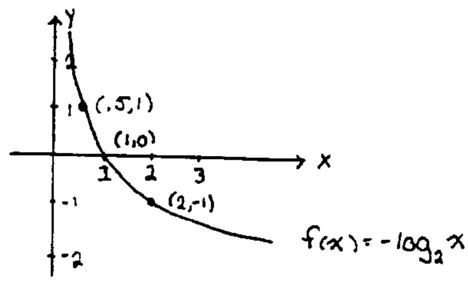

Figure 7.

Example 17. Sketch the graph of $g(x) = -\log_2(-x)$. What is the domain of g?

Since we can take the log of positive numbers only, the domain of this function is $x < 0$. Notice that $g(x) = f(-x)$, so the graph of g can be obtained by reflecting the graph of f from the previous example over the y-axis.

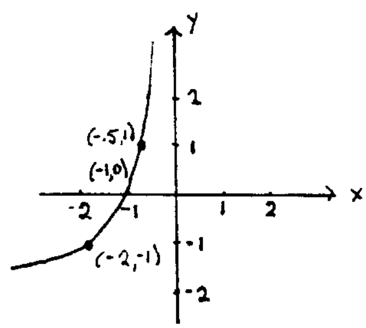

Figure 8.

Example 18. Sketch the graph of $h(x) = \log_3 27x^2$.

What is the domain of h?

We can take the log of positive numbers only, so the domain of h is all nonzero x. Let's use what we know about logs to rewrite $h(x)$ so that it looks more familiar.

$h(x) = \log_3 27x^2 = \log_3 27 + \log_3 x^2 = 3 + \log_3 x^2.$

$h(x)$ is an even function (because $h(-x) = h(x)$), so its graph is symmetric about the y-axis. Therefore, if we graph $y = 3 + 2\log_3 x$ and reflect this graph about the y-axis, we'll be all set.

How do we graph $y = 3 + 2\log_3 x$? We'll start with the graph of $\log_3 x$ and build it up from there.

$\log_3 x$ and 3^x are inverse functions, so their graphs are reflections about the line $y = x$.

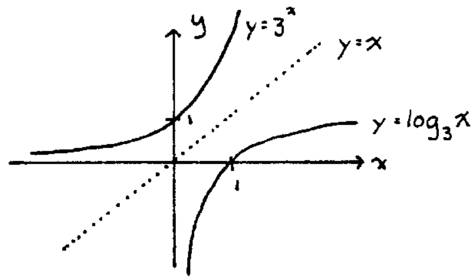

Figure 9.

Multiplying $\log_3 x$ by 2 stretches the graph vertically, and adding 3 shifts the graph up 3 units. The graph of $y = 3 + 2\log_3 x$ is shown below.

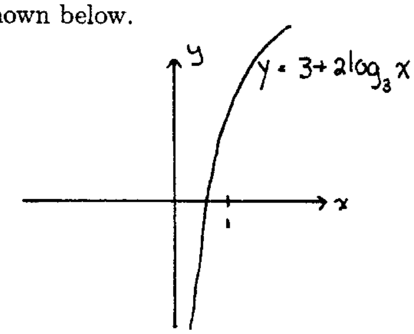

Figure 10.

Should we want the exact value of the x intercept we can get it by solving the equation $0 = 3 + 2\log_3 x$. We obtain

$$\begin{aligned} \log_3 x &= \frac{-3}{2} && \text{Exponentiate.} \\ 3^{\log_3 x} &= 3^{-3/2} \\ x &= \frac{1}{3^{3/2}} = \frac{1}{3\sqrt{3}} \end{aligned}$$

The graph of $h(x)$ is given below.

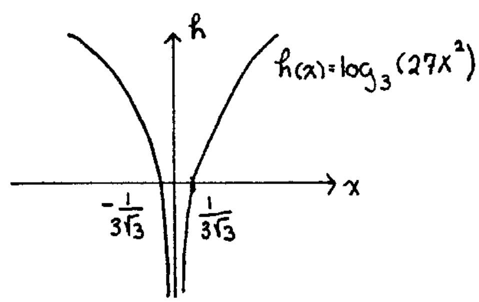

Figure 11.

Answers to Exercises 1 and 2:

1. a) $3^4 = 81$ b) $2^{-.5} = \frac{1}{\sqrt{2}}$ c) $\log_w A = b$

2. a) $y = \log_6 40 \Leftrightarrow 6^y = 40.$

   Since $6^2 = 36$ and $6^3 = 216$, we know that $2 < \log_6 40 < 3$

   b) $y = \log_{10} 3789 \Leftrightarrow 10^y = 3789.$

   Since $10^3 = 1,000$ and $10^4 = 10,000$, we know that $3 < \log_{10} 3789 < 4$

   c) $y = \log_2 40 \Leftrightarrow 2^y = 40.$

   Since $2^5 = 32$ and $2^6 = 64$, we know that $5 < \log_2 40 < 6$

12.7 Problems For Chapter 12

1. Sketch the graphs of $f(x) = 2^x$ and the graph of $\log_2 x$ on the same set of axes. Label **three** points on each graph.

2. Fill in the blanks: When we write $\log_2 3$ we say "log base two of 3". We mean the power to which 2 must be raised in order to get 3.

   (a) When we write $\log_5 14$ we say "_______". We mean the power to which ___ must be raised in order to get ___.

   (b) When we say "log base 4 of 8" we write ___. We mean the power to which ___ must be raised in order to get ___.

   (c) We mean the power to which e must be raised in order to get 5, so we write _______ and we say _______.

3. Approximate the values of the following logarithms by giving two consecutive integers, one of which is a lower bound and the other an upper bound for the expressions given. Do this without a calculator. (You can use the calculator to check your answers, but the idea of the problem is to get you to think about what logarithms mean.) Explain your reasoning as in the example below.

   Example: $\log_{10} 113$ is between 2 and 3

   Reasoning: $\log_{10} 113$ is the number we must raise 10 to in order to get 113
   $10^2 = 100$ and $10^3 = 1000$

   $a)\ \log_7 50$ $e)\ \log_2 \sqrt{30}$
   $b)\ \log_{10}(.5)$ $f)\ \log_5 \sqrt{30}$
   $c)\ \log_{10}(.05)$ $g)\ \log_{10} \sqrt{30}$
   $d)\ \log_3 29$

4. Simplify the following: We'll let $\log x$ be shorthand for $\log_{10} x$.

   $a)\ 3^{\log_3 2}$ $f)\ 10^{-\log x}$
   $b)\ \log x + \log x^2 - 3\log x$ $g)\ 10^{-.5\log x}$
   $c)\ 2\log(x+3) - 3\log(x+3) + \log(10^{\sqrt{7}})$ $h)\ 3^{-\log_3(x+y)}$
   $d)\ 10^{\log x^2}$ $i)\ 2^{(\log_2 10 - \log_2 5)}$
   $e)\ 10^{3\log x}$ $j)\ 10^{\frac{\log x}{2}}$

5. Simplify the following: (No calculators, except to check your answers if you like.)

   $a)\ \log_2 \sqrt{8}$ $d)\ \log_3(1/9)$ $g)\ \log_k(k^x k^y)$
   $b)\ \log_{10} .001$ $e)\ \log_k k^{3x}$
   $c)\ \log_2\left(\frac{4}{\sqrt{8}}\right)$ $f)\ \log_k 1$

6. If $\log_2 u = A$ and $\log_2 w = B$, express the following in terms of A and B (eliminating u and w).

   $a)\ \log_2(u^2 w)$ $c)\ \log_2(1/\sqrt{w}))$
   $b)\ \log_2(u^3/w^2)$ $d)\ \log_2(\frac{2}{\sqrt{uw}})$

7. Solve for $t : P = P_0 e^{kt}$.

8. (a) Use your calculator to help you evaluate the following limits:

$$i) \lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} \qquad ii) \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}.$$

   (b) Which grows faster as $x \to \infty, \ln x$ or $\sqrt{x}$?

9. Sketch a rough graph of $y = \ln x - \ln(x^3) + 4\ln(x^2)$. (Hint: this will be straightforward after you have rewritten this in the form $y = K \ln x$, where K is a constant.

10. On the same set of axes sketch the graphs of $y = \ln x$ and $y = 3\ln x$. Label the x-intercept. Note that the latter is the graph of $y = \ln x^3$.

11. Solve for x: (Don't expect 'pretty' answers!)

$a) 10^{2x} = 93$	$e) \ln x + 2 = 5$
$b) 10^{3x+2} = 1,000,000$	$f) \log_{10} x = 17$
$c) 2^{x+1} = 7$	$g) \ln(5x - 40) = 3$
$d) 3^x 3^{x^2} = 3$	$h) \log_{10}(2x^2 + 4) = 2$

12. (a) Approximate $\log_3 16$ (with error less than 0.005) using your calculator.
    (b) Rewrite $\log_3 16$ in terms of log base 10.
    (c) Rewrite $\log_3 16$ in terms of log base e.
    (d) Rewrite $\log_3 16$ in terms of log base 7.

13. Solve for x (some of your answers will not be pretty - but persevere.)
    (a) $3\ln x + 5 = (\ln x)\ln 2$
    (b) $2\left(7^{1+\log x}\right) = 8$
    (c) $Ke^x + K = Le^x - L$, where K and L are constants and $0 < K < L$.
    (d) $R(1+n)^{nx} = (Pn)^x$, where P, R and n are constants.
    (e) $3b^x = c^x 3^{2x}$, where b and c are constants.

14. Solve for x.

$a) \quad 2^{x^2} 2^x = 3^x$	$b) \quad 3^{x^2+2x} = 1$
$c) \quad 3\ln(x^4) - 2\ln 2x = 10$	$d) \quad e^{2x} + e^x - 6 = 0$
$e) \quad e^x + 8e^{-x} = +6$	$f) \quad (\ln x)(\ln 5) = \ln 4x$

15. The Richter scale, introduced in the mid-1900's, measures the intensity of earthquakes. A measurement on the Richter scale is given by

$$M = \log \frac{I}{\mathrm{S}}$$

where I is the intensity of the quake and S is some standard.

Suppose we want to compare the intensity, I_1, of a particular earthquake with intensity I_2 of a less violent quake. The difference in their measurements on the Richter scale is

$$\log \frac{I_1}{S} - \log \frac{I_2}{S} = \log \left[\frac{\frac{I_1}{S}}{\frac{I_2}{S}} \right] = \log \frac{I_1}{I_2}.$$

In particular, suppose that one earthquake measures 7 on the Richter scale and another measures 4. Then

$$\log \frac{I_1}{I_2} = 7 - 4 = 3.$$

Therefore, $\frac{I_1}{I_2} = 10^3 = 1000$. The former earthquake has 1000 times the intensity of the latter.

a) On August 20th, 1999 there was an earthquake in Costa Rica (50 miles south of San Jose) measuring 6.7 on the Richter scale and another in Montana (near the Idaho border) measuring 5 on the Richter scale. How many times more intense was the Costa Rican earthquake?

b) The 1989 earthquake in San Francisco measured 7.1 on the Richter scale. How many times more intense was the earthquake in Turkey on August 17th 1999 measuring 7.4 on the Richter scale?

16. Acidity is determined by the concentration of hydrogen ions in a solution. The pH scale, proposed by Sorensen in the early 1900s, defines pH to be $-\log[H^+]$ where $[H^+]$ is the concentration of hydrogen ions given in moles per liter. A pH of 7 is considered neutral; a pH greater than 7 means the solution is basic, while a pH of less than 7 indicates acidity.

    (a) If the concentration of hydrogen ions in a solution is increased tenfold, what happens to the pH?

    (b) If a blood sample has a hydrogen ion concentration of 3.15×10^{-8}, what is the pH?

    (c) You'll find that the blood sample described in part (b) is mildly basic. Which has a higher concentration of hydrogen ions: the blood sample or something neutral? How many times greater is it?

17. The "Rule of 70" says that if a quantity grows exponentially at a rate of $r\%$ per unit of time then its doubling time is usually about $70/r$. This is merely a rule of thumb. Now we will determine how accurate an estimate this is and for what values of r it should be applied.

    Suppose that a quantity Q grows exponentially at $r\%$ per unit of time t. Thus, $Q(t) = Q_0\left(1 + \frac{r}{100}\right)^t$.

    (a) Let $D(r)$ be the doubling time of Q as a function of r. Find an equation for $D(r)$.

    (b) On your graphing calculator, graph $D(r)$ and $70/r$. For what values of r would you say that the Rule of 70 provides a good approximation?

18. Where does the "Rule of 70", referred to in Problem 17, come from? In this problem we investigate the rule.

    (a) Use the equation $Q(t) = Q_0\left(1 + \frac{r}{100}\right)^t$ and solve for the doubling time.

    (b) For r small, $\ln\left(1 + \frac{r}{100}\right)$ can be well approximated by the tangent line to $\ln x$ at $x = 1$. Explain this assertion and give the tangent line approximation.

    (c) In your answer to (a), replace $\ln\left(1 + \frac{r}{100}\right)$ by its tangent line approximation. Explain how to get from this expression to the "Rule of 70".