Math Xa
First Exam: Annotated Solutions

1 a) • (i)
• (ii)

b) • (i)
• (ii)

c) • (iv)
• (ii)
• (i)

Comments

a) f is positive, increasing and concave down
- positive → No info about f'
- increasing → $f' > 0$
- concave down → f' decreasing

b) g is negative, increasing and concave down
- negative → No inf about g'
- increasing → $g' > 0$
- concave down → g' decreasing

where a function is increasing, its derivative is positive
where a function is decreasing, its derivative is negative
where a function is concave down, its derivative is decreasing
where a function is concave up its derivative is increasing

c) h' is negative, increasing, and concave down
- negative → h decreasing
- increasing → h concave up
- concave down → Not of much interest to us - no basic inf about h.

2. a)

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{5(3+h)}{2(3+h)+4} - \frac{5\cdot 3}{2\cdot 3+4}}{h}$$

$$= \lim_{h\to 0} \left[\frac{15+5h}{6+2h+4} - \frac{15}{10}\right]\cdot\frac{1}{h}$$

Common Error: $\frac{A}{h} \neq A\cdot\frac{h}{1}$; $\frac{A}{h} = A\cdot\frac{1}{h}$

$$= \lim_{h\to 0} \left[\frac{15+5h}{10+2h} - \frac{3}{2}\right]\cdot\frac{1}{h}$$

Get a common denominator

$$= \lim_{h\to 0} \left[\frac{15+5h}{2(5+h)} - \frac{3(5+h)}{2(5+h)}\right]\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{15+5h-15-3h}{2(5+h)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{2h}{2(5+h)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{1}{5+h} = \frac{1}{5} = .2$$

b) Average rate of change of f over $[2.99, 3]$ $= \dfrac{f(3)-f(2.99)}{3-2.99} = \dfrac{\frac{5\cdot 3}{2\cdot 3+4} - \frac{5(2.99)}{2(2.99)+4}}{.01} = \dfrac{\frac{15}{10} - \frac{14.95}{9.98}}{.01} \approx .2004008\ldots$

Comment: Your answer to (b) should be close (very close) to your answer to (a). If they aren't close you need to look for an error.
Recall: def'n: Average rate of change of f over $[a,b] = \dfrac{f(b)-f(a)}{b-a}$. Some people made this unnecessarily difficult!

3. a) at $t=3$

graph of T ↗ → ↘
sign of T' + 3 −

b) When is T' greatest? Answer: at $t=-1$.

c) When is T' most negative? Answer: at $t=4$.

d) On $[7,8]$ T' is negative so T is decreasing. T' is a negative constant, so T is decreasing at a constant rate.

Common Errors

(a) T is greatest at $t=3$.
The greatest value of T is $T(3)$.

(b,c) The answers should be a value of t, NOT an interval.
For instance, at $t=4$, the graph of T is horizontal. It is NOT increasing rapidly.
T' increasing doesn't imply T increasing!!

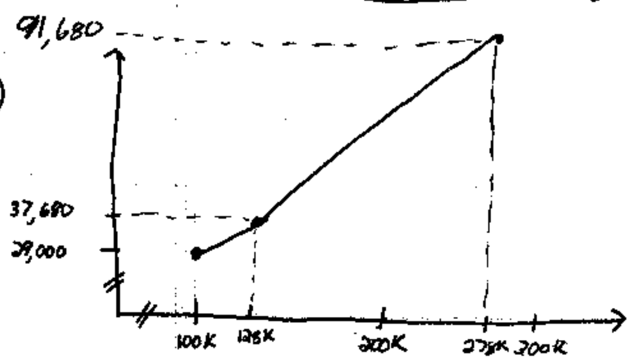
4

a) $T(x) = \begin{cases} 29{,}000 + .31(x - 100{,}000) & 100{,}000 \le x < 128{,}000 \\ 29{,}000 + .31(28{,}000) + .36(x - 128{,}000) & 128{,}000 \le x \le 278{,}000 \end{cases}$

NOTE: there are a number of different combinations of $<$ + $\le$ that correctly model the function

COMMON ERRORS: Some people didn't model the tax rates as marginal rates, i.e. they wrote equations where the rate applied to the entire income, not just income over the breaking points.

b.)



c) $T(200{,}000) = 37{,}680 + .36(200{,}000 - 128{,}000)$
$= 63{,}600$

This is more than twice the tax paid by someone earning \$100,000

COMMON ERRORS: starting the graph at 0 instead of 100K
making the 2nd piece less steep than the first piece

d. $T'(120{,}000) = .31$. This means that the tax rate for an income of \$120K is 31% + the next dollar earned will ~~[illegible]~~ mean an additional \$.31 in taxes owed

e. $T'(150{,}000) = .36$. At an income of \$150K, the tax rate is 36%. I.e., the next dollar earned will mean an additional \$.36 in taxes owed.

COMMON ERRORS: Some people described T' as the "instantaneous rate of change" but did not interpret what this meant in practical terms.

5. a) VIII
   b) VI
   c) IV
   d) II.

a) Common Error: for large x the value of the function appproaches a positive number, but the slope approaches zero.

6. a) $M_1(t) = M_0 2^{t/7}$

$M_2(t) = M_0 (1.01)^{12t}$

$M_3(t) = M_0 3^{t/10}$

Common Error: In account 2 - the money increases by 1% every month - That's different from 1% per year compounded monthly. The former is $M_0(1.01)^{12t}$ and the latter is $M_0\left(1+\frac{.01}{12}\right)^{12t}$. The first is a better deal!

b) $M_1(t) = M_0 (2^{1/7})^t \approx M_0 \cdot 1.10408^t$

$M_2(t) = M_0 \cdot (1.01^{12})^t \approx M_0 (1.12682)^t$

$M_3(t) = M_0 (3^{1/10})^t \approx M_0 (1.116123)^t$

The effective annual interest rates are:

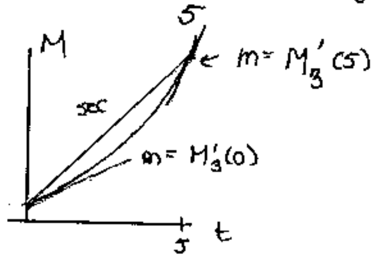
$M_1: \approx 10.4\%$

$M_2: \approx 12.68\%$

$M_3: \approx 11.61\%$

So Bank 2 gives the most favorable rate and Bank 1 gives the least favorable rate.

c) $M_3'(0) < \frac{M_3(5)-M_3(0)}{5} < M_3'(5)$



M_3 is an exponential function, so it is increasing at an increasing rate. Therefore, the slope at $t=0$ is less than the slope at $t=5$. The slope of the secant line drawn corresponds to the average rate of change, $\frac{M_3(5)-M_3(0)}{5}$. It is greater than the slope at 0 and less than the slope at 5.

Common Errors

① $M_3'(0)$ is Not zero - it's the slope at $t=0$

② $M_3'(0)$ is the slope when $t=0$, NOT when $M=0$.

③ $\frac{M_3(5)-M_3(0)}{5}$ is NOT the average of $M_3'(0)$ and $M_3'(5)$. Consider $s(t)$ = position & s' = velocity. The average velocity from $t=a$ to $t=b$ is $\frac{\Delta s}{\Delta t} = \frac{s(b)-s(a)}{b-a} \neq \frac{s'(b)-s'(a)}{2}$. For instance, your trip could start and end in a parked car - with velocity zero. And yet your ave. vel. can be positive!

7. Reupholstering rates:

Chairs: $\frac{8 \text{ hrs}}{C \text{ chairs}} = 8/C$ hrs/chair

couches: $\frac{D \text{ hours}}{1 \text{ couch}} = D$ hours/couch

painting: $S \frac{\text{sq.ft.}}{\text{hr.}} \rightarrow 1/S \frac{\text{hrs.}}{\text{sq. Foot}}$

time to paint:

A sq.ft $\longrightarrow A \text{ sq.ft } \frac{1}{S} \frac{\text{hrs}}{\text{sq.ft.}} = \frac{A}{S}$ hours

X chairs $\longrightarrow X \text{ chairs} \cdot \frac{8}{C} \frac{\text{hrs.}}{\text{chair}} = \frac{8X}{C}$ hours

Y couches $\longrightarrow Y \text{ couches} \cdot D \frac{\text{hrs.}}{\text{couch}} = YD$ hours

total time: $\frac{A}{S} + \frac{8X}{C} + YD$ hours.

COMMON ERROR: dividing instead of multiplying or vice versa. Checking units is a good way to avoid this pitfall

b) Hours needed for Q chairs:

$Q \text{ chairs} \cdot \frac{8}{X} \frac{\text{hrs.}}{\text{chair}} = \frac{8Q}{X}$ hours.

Hours already spend: T hours

Hours remaining: $\frac{8Q}{X} - T$ hours