Answers to the 1st Exam
Math Xa 1997

1. a) $f(c)$ because f is increasing on (a,c): $f'>0$ on (a,c)

 b) $f(e)$ is larger because the slope of f is negative between e & g; therefore f is decreasing

 c) $f(d)$ is greatest because the graph of f is increasing to this point (to d), and decreasing afterwards.

 d) No. the graph of f' only shows the slope of f. Because we don't know a starting point we don't know the largest value of f.

 e) at $x=e$ because f' is least there.

 f) i.e. where is $f'\le 0$ and f' decreasing? $[d,e]$, $[g,h]$

 g) at $x=a$ f' is decreasing so f is concave down.

2. a) $(-\infty,-2)$ & $(0,2)$ & $(4,\infty)$

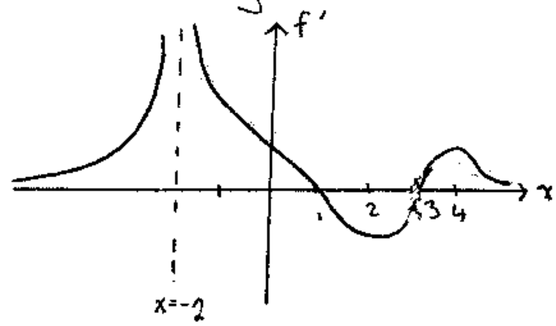
 b) $(1,3)$

 c) $(1,3)$

 d) where f is concave down: $(-2,2)$ & $(4,\infty)$

 e) where f is increasing and concave down $(-2,1)$ & $(4,\infty)$

 f)



3. [see next page]

4. a) $S(0)-S(-2)$ miles

 b) $\frac{\Delta \text{position}}{\Delta \text{time}} = \frac{S(4)-S(2)}{2}$ $\frac{\text{miles}}{\text{hr.}}$

 c) $S'(4)$ mph

 d) $|S'(4)|$ mph

 a) $S'(5)=-30$

 b) $S(9)=+20$

5. done in the review session

6. a) $B(t)=\begin{cases}1000(1.10)^t & \text{for } 0\le t\le 3\\ 1000(1.10)^3+60(t-3) & \text{for } t>3\end{cases}$

 b) $B(5)=1000(1.10)^3+60(5-3)=1451$ beavers

 c) $B'(5)=60$
 $B(t)$ is a line for $t>3$

7. a) At a temperature of 60°F, 2030 people are expected to attend.

 b) When the temperature is 55°F a 1 degree increase in temperature means that approximately 40 more people will expected at the concert.

8) [Another page]

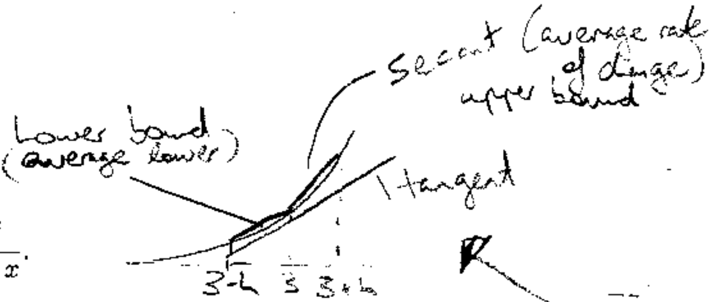
9) a) $Q(t)=700\cdot 3^{t/4}$ b) $Q(t)=700(.006)^t$

 c) $Q(t)=700\cdot\left(\frac{1}{2}\right)^{4t}$

 d) For each acceptable value of x there is exactly one y-value.

3. (14 points) Let

$$f(x) = \frac{x^2}{1+x}.$$



(a) Use numerical methods to approximate $f'(3)$.

At $x = 3$ the graph of f is increasing and concave up. Give one approximation that is an upper bound for $f'(3)$, and another that is a lower bound for $f'(3)$.

Explain briefly how you *know* one is an upper bound and the other is a lower bound. (We suggest that you use a picture to explain this.)

$$f'(3) \approx \frac{\frac{(3+.0001)^2}{1+3+.0001} - \frac{3^2}{1+3}}{.0001}$$

$$\approx \frac{\frac{(3.0001)^2}{4.0001} - \frac{9}{4}}{.0001}$$

Upper Bound — Slope of secant $\approx .937501562$ ✓

$$f'(3) \approx \frac{\frac{(3-.0001)^2}{1+3-.0001} - \frac{3^2}{1+3}}{-.0001}$$

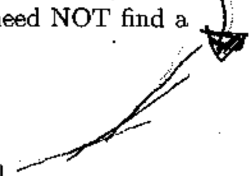
$$\approx .93749837$$

$\boxed{\text{Lower Bound}}$ ✓

with positive h

Excellent!!

Because the curve is concave up the secant line must be the upper bound and the secant line with the negative h must be the lower bound.



curve concave up all the tangents are lower

(b) Using the limit definition of derivative, find $f'(3)$ exactly. You must produce an exact answer, not a numerical approximation. All your algebra must be explicitly demonstrated. (You need NOT find a general formula for $f'(x)$, only $f'(3)$.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \checkmark$$

$$f'(3) = \lim_{h \to 0} \frac{\frac{(3+h)^2}{1+3+h} - \frac{3^2}{1+3}}{h} \checkmark$$

$$= \lim_{h \to 0} \left(\frac{(3+h)(3+h)}{4+h} - \frac{3^2}{4} \right) \times \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{9+6h+h^2}{4+h} - \frac{9}{4} \right) \times \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{36}+24h+4h^2-\cancel{36}-9h}{16+4h} \right) \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{15h+4h^2}{16+4h} \times \frac{1}{h} \qquad h \neq 0$$

perfect

$$15/16 \checkmark$$

(c) Given your answer to (a), how much confidence do you have in your answer to (b)?

A lot, because 15/16 is between the upper and lower bounds found in part a. ✓

8. (12 points) Let $C(t)$ be the value of a car as a function of the number of years since it was purchased. When purchased 2 years ago it cost \$12000, and its value now is \$8670.

(a) Assuming $C(t)$ is linear, find an equation for $C(t)$.

$C(0) = 12000$, $C(2) = 8670$

average rate of change $= \dfrac{8670 - 12000}{2 - 0}$

$= -1665$ per yr.

$C(t) =$ $C_0 - (1665 \times t)$ ← C_0 could be replaced by \$12000 Yes, do it.

(b) Assuming $C(t)$ is exponential, find an equation for $C(t)$.

$C(0) = 12000$
$C(2) = 8670$

$C(t) = C_0 \times (.85)^t$

$C(t) = C_0\, a^t$
$C(2) = C_0\, a^2$
$8670 = 12000 \times a^2$
$.7225 = a^2$
$(.7225)^{1/2} = (a^{1/2})^2$
$a = .85$

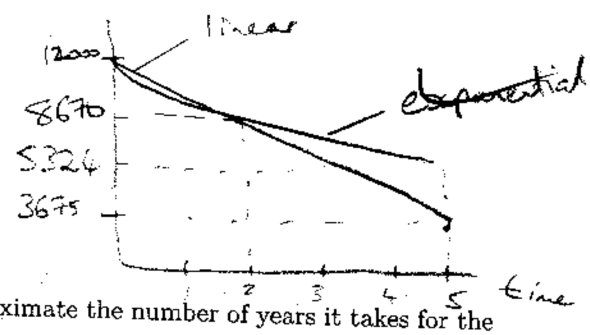
$C(t) =$ $C_0 \times (.85)^t$ ← C_0 could be replaced by \$12000 Yes.

Each year the value of the car decreases by 15 %.

(c) Which model (linear or exponential) predicts a higher value for the car 5 years from now? Explain your answer with a sketch.

$C_0 - (1665 \times 5) = C_0 \times (.85)^5$
$3675 = 5324.46$

The exponential value is higher
Because we take .85 of a decreasing number



(d) Using the **exponential** model (and your calculator), approximate the number of years it takes for the value of the car to be half of its original sale price.

$C(t) = C_0 \times (.85)^t$
$\tfrac{1}{2} C_0 = C_0 \times (.85)^t$
$6000 = 12000 \times (.85)^t$
$0 = .85^t - \tfrac{1}{2}$
$= 4.265$

$C(t) = 12000 \times (.85)^{4.265}$
very close to 6000

So the value of the car is half the original sale price after just over 4¼ yrs.

11. (11 points) A radioactive quantity decays exponentially. When there are 80 grams of the radioactive substance, the amount is decreasing at a rate of 4 grams per day. At what rate is the amount decreasing when there are 37 grams of the substance?

Hint: There is no need to find an equation for the number of grams of radioactive material at time t. Think instead about the rate at which exponential functions change!

$R(t) = R_0 \times a^t$

$R'(80) = -4$

$R'(37) = ?$

$R'(t) = k \times R(t)$

$-4 = k \times 80$

$k = -.05$

So $R'(t) = -.05 \times 37$

$= -1.85$ ✓

Amount is decreasing at a rate of 1.85 grams per day when there are 37 grams of the substance ✓