Annotated Solutions to the
Final Examination: Jan. 21, 2000
MATH Xa

1\. a) $s(0) = \boxed{4 \text{ ft.}}$

b) $s'(t) = -32t + 48$

$s'(0) = -32(0) + 48 = \boxed{48 \text{ ft/sec.}}$

c) $0 = -32t + 48$

$t = \frac{48}{32} = \boxed{1.5 \text{ seconds}}$

d) $s(1.5) = -16(1.5)^2 + 4(1.5) + 4$

$= \boxed{40 \text{ ft.}}$

e) $m_{sec} = \frac{s(1.5) - s(0)}{1.5 - 0} = \frac{40-4}{1.5}$

$= \boxed{24 \text{ ft/sec}}$

Note: Ave. vel $= \frac{\Delta \text{position}}{\Delta \text{time}}$

Common Errors

- $\frac{\Delta \text{vel}}{\Delta \text{time}}$ = ave accel. Not ave. vel.
- You can't average the final & initial velocities! Suppose a trip begins & ends at rest (with velocity = 0). The average velocity will NOT be zero. That would ignore the trip.

f) Find when it hits the ground. Then evaluate s' at this t-value.

$-16t^2 + 48t + 4 = 0$

$-4t^2 + 12t + 1 = 0$

$4t^2 - 12t - 1 = 0$

$t = \frac{12 \pm \sqrt{144 + 16}}{8} = \frac{12 \pm \sqrt{160}}{8}$

$= \frac{3}{2} \pm \frac{4\sqrt{10}}{8} = \frac{3}{2} \pm \frac{\sqrt{10}}{2}$

We want $\frac{3}{2} + \frac{\sqrt{10}}{2}$

$s'\left(\frac{3}{2} + \frac{\sqrt{10}}{2}\right) = -32\left(\frac{3 + \sqrt{10}}{2}\right) + 48$

$= -48 + 48 - 16\sqrt{10} = \boxed{-16\sqrt{10} \text{ ft/sec.}}$

Notice that the velocity is negative, as expected.

Common Error:
You can't say the vel. is −48 ft/sec by symmetry, because the object started 4 ft. in the air. Therefore, by the time it hits the ground the vel. should be more negative than this. Indeed, the answer is ≈ −50.596 ft/sec.

g) $|s'(1)| = |-32(1) + 48| = \boxed{16 \text{ ft/sec}}$

h) $\boxed{\text{Yes - at } t = 2}$ (Now we can use symmetry)

or: $-16 = -32t + 48$ set $s' = \pm 16$

$-64 = -32t$

$t = 2.$

2a. $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \frac{\left(\frac{2+h-1}{2+h+3}\right) - \left(\frac{2-1}{2+3}\right)}{h}$

$= \lim_{h \to 0} \frac{\frac{h+1}{h+5} - \frac{1}{5}}{h}$

$= \lim_{h \to 0} \left[\frac{5h+5 - (h+5)}{(h+5)5}\right] \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{5h+5-h-5}{(h+5)5} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{4\cancel{h}}{5(5+h)} \cdot \frac{1}{\cancel{h}}$ cancel because $h \neq 0$

$= \lim_{h \to 0} \frac{4}{5(5+h)}$

$= \frac{4}{25}$

b) Method 1: Use quotient or product rule

$f(x) = \frac{x-1}{x+3} \Rightarrow f'(x) = \frac{(x+3)(1) - (x-1)(1)}{(x+3)^2}$

$f'(x) = \frac{x+3-x+1}{(x+3)^2} = \frac{4}{(x+3)^2}$

$f'(2) = \frac{4}{5^2} = \frac{4}{25}$ Answers agree

Method 2: Numerical Approx.

$(2, f(2))$ & Nearby pt. $(2.001, f(2.001))$

$m_{tan} \approx \frac{f(2.001) - f(2)}{.001} = \frac{\frac{2.001-1}{5.001} - \frac{1}{5}}{.001} = \frac{\frac{1.001}{5.001} - \frac{1}{5}}{.001}$

$\approx .159968$

$\frac{4}{25} = \frac{16}{100} = .16$ so the answers are very close - the $\frac{4}{25}$ is correct.

3\. Plan: 1. Find f'' & set $= 0$ to find pt. of inflection (it's a poly, so f'' isn't undefined)
2. Verify that concavity changes at Point of inflection
3. Use f to find y-coord of pt. of infl. $f(2)$
4. Use f' to find slope of tangent at $x = 2$
5. Use pt.-slope eq'n of line to find eq'n of tangent.

$f(x) = x^3 - 6x^2 + 3$

$f'(x) = 3x^2 - 12x$

$f''(x) = 6x - 12$

(1) $f''(x) = 0 \Rightarrow 6x - 12 = 0$

$x = 2$

(2) sign of f'': $-$ | $x = 2$ | $+$

(3) $f(2) = (2)^3 - 6(2^2) + 3$

$= 8 - 24 + 3$

$= -13$

pt. of inflection: $(2, -13)$

(4) $f'(2) = 3(2)^2 - 12(2)$

$= -12$

slope of tangent $= -12$

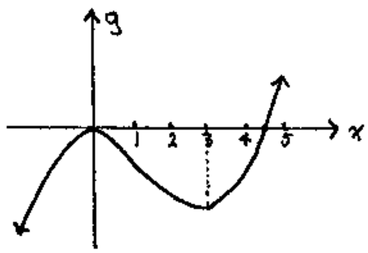
(5) $y - y_1 = m(x - x_1)$

$y + 13 = -12(x - 2)$

or $y = -12x + 24 - 13$

$\boxed{y = -12x + 11}$

4. a.



Because the zeros of g are 0 & 4.5
the x-intercepts are 0 & 4.5

Because $g' = 0$ at 0 & 3 there's a horizontal tangent line at 0 & 3

Because the leading coefficient is 2 & deg = 3
the graph looks like $\swarrow$ ..... $\nearrow$

Because the poly. is of degree 3, one of the zeros must have multiplicity 2 (because there are only 2 of them)

b. $f(x) = \dfrac{g(x)}{h(x)}$

i) The x-intercepts of f are the zeros of g
$\boxed{x = 0,\ x = 9/2}$

ii) The vertical asymptotes are at the zeros of h. $\boxed{x = 1,\ x = 3}$

iii) For x very large in magnitude
f(x) looks like $\dfrac{2x^3}{-x^4}$ (the leading terms)

so f(x) looks like $\dfrac{2}{-x} \to 0$ as $x \to \pm\infty$
$\boxed{y = 0}$ is the horizontal asymptote.
(The degree of the numerator is less than the degree of the denominator, so the x-axis is the horizontal asymptote.)

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5. a. $y = \frac{1}{5}(x^3 + \sqrt{e})^{\pi}$

$y' = \frac{\pi}{5}(x^3 + \sqrt{e})^{\pi - 1} \cdot 3x^2$

Errors: $\sqrt{e}$ is a constant: its derivative is 0.
Notice that the quotient rule is totally unnecessary

b. $y' = 3 \cdot e^{(g(x))^2} \cdot 2g(x) \cdot g'(x)$

c. $y' = f'(\ln x) \cdot \frac{1}{x} - \frac{1}{f(x)} \cdot f'(x)$

or $\dfrac{f'(\ln x)}{x} - \dfrac{f'(x)}{f(x)}$

d) $y = [f(x) \cdot g(3x)]^{1/2}$

$y' = \frac{1}{2}[f(x) \cdot g(3x)]^{-1/2} \cdot [f'(x)\, g(3x) + f(x)\, g'(3x) \cdot 3]$

e) Rewrite: $y = \ln x + \ln(f(x)) - \ln(3x^3 + 2)^{1/2}$
$y = \ln x + \ln(f(x)) - \frac{1}{2}\ln(3x^3 + 2)$

so $y' = \frac{1}{x} + \dfrac{f'(x)}{f(x)} - \frac{1}{2}\dfrac{1}{3x^3 + 2} \cdot 9x^2$

Most people wasted effort by not taking advantage of log rules.

Math X solutions to final, Jan 21^{st} 2000

(6) (a) $[-2,-1)$ and $(3,4]$

(b) $g(x) = [f(x)]^4$, so by chain rule,
$$g'(x) = 4[f(x)]^3 f'(x)$$

(c) critical points for $g(x)$ – where $g'=0$ undefined, or endpoint of domain.
$g'(x)=0$ when either $f(x)=0$ or $f'(x)=0$
so critical points are $-2,-1,0,1,3,4,6$

(d) $g'(x)$ is positive when either both $f(x)$ and $f'(x)$ are positive, or when they're both negative. This works out to $(-1,0)$, $(1,3)$ and $(4,6)$

(e) local maxima for g at $x=0$ and 3

(f) local minima (also absolute minima) at $x=-1$, 1 and 4

(g) yes, has abs. max when $x=6$, $g(6)=81$

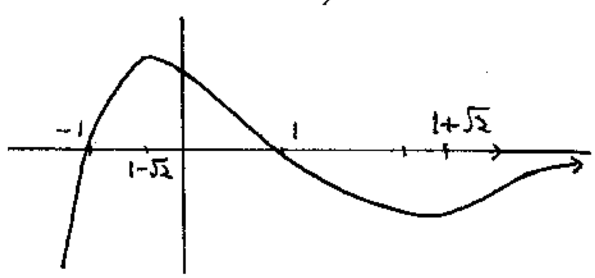
(7) $f(x) = e^{-x}(-x^2+1)$

(a) $f(x)=0$ just when $-x^2+1=0$, since $e^{-x} \neq 0$, so x-intercepts are at $x=\pm 1$

(b) $f(x)$ has no endpoints, so critical points are when $f'(x)=0$
$$f'(x) = e^{-x}(-2x) + (-x^2+1)(-e^{-x})$$
$$= e^{-x}(x^2-2x-1)$$
again $e^{-x} \neq 0$, so $f'(x)=0$ only when $x^2-2x-1=0$
$$\text{or} \quad x = \frac{2 \pm \sqrt{2^2+4}}{2} = 1 \pm \sqrt{2}$$

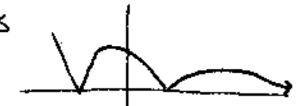
Checking sign of $f'(x)$ reveals $x = 1-\sqrt{2}$ is a maximum and $x = 1+\sqrt{2}$ is a (local) minimum

(c)



Solutions continued

(7)(d) $g(x) = |f(x)|$ looks like:



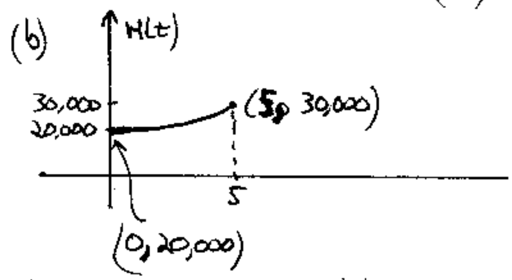
(i) critical points are same as for $f(x)$ plus new critical points at $x=-1$, and 1, so total list of critical points is $-1, 1-\sqrt{2}, 1, 1+\sqrt{2}$

(ii) 2^{nd} derivative test cannot be used when $x=-1$ and 1 as $g'(x) \neq 0$ at these two points

(8) (a) $H(t) = 20{,}000\, b^t$

we know $H(5) = 30{,}000$
$$= 20{,}000 \cdot b^5$$
$$\text{so} \quad b^5 = \frac{30{,}000}{20{,}000} = \frac{3}{2}$$
$$b = \left(\frac{3}{2}\right)^{1/5}$$
$$\text{so} \quad H(t) = 20{,}000 \cdot \left(\frac{3}{2}\right)^{t/5}$$

(b)



(c) $H(t) = 20{,}000 \cdot \left(\frac{3}{2}\right)^{t/5}$ is an exponential growth function which grows without bound as $t \to \infty$. In this case, however there can only be 40,000 households with computers so $H(t)$ cannot rise above 40,000, and so cannot continue as an exponential function. In fact $H(t)$ cannot continue to be concave up for too much longer, and must eventually become concave down, so $H''(t)$ will change from positive to negative.

(d) This increasing function implies that $H'(t)$ will always be positive.

Solutions continued

(9) (a) when $t=3$ (noon) then

$$10+10\log_{10}(3^2+1) = 10+10\log_{10}(10)$$
$$= 10+10\cdot 1 = 20$$

(b) rate of change of value is

$$f'(t) = 10\,\frac{1}{\ln(10)\cdot(t^2+1)}\cdot(2t)$$
$$= \frac{20t}{\ln(10)(t^2+1)}$$

then $f'(3) = \dfrac{20\cdot 3}{(\ln 10)(10)} = \dfrac{6}{\ln 10}$

(c) $f'(t)$ is at a maximum ("increasing most rapidly") when $f''(t)=0$

$$f''(t) = \left(\frac{20}{\ln 10}\right)\cdot\frac{(t^2+1)-t(2t)}{(t^2+1)^2}$$
$$= \left(\frac{20}{\ln 10}\right)\cdot\frac{1-t^2}{(t^2+1)^2}$$

this will equal zero when $1-t^2=0$ or when $t=\pm 1$. In our model the only time that works is for $0\le t\le 8$, so increases most rapidly when $t=1$, or at 10am

(d) Solve $25 = 10+10\log_{10}(t^2+1)$

$$15 = 10\log_{10}(t^2+1)$$
$$1.5 = \log_{10}(t^2+1)$$
$$10^{1.5} = t^2+1$$

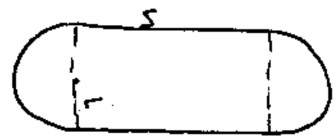
so $t = \sqrt{10^{1.5}-1}$

≈ 5.5338 hours

or 332 minutes after 9am,

which is about 2:32 pm

(10)



let r = radius of semicircles,
s = length of straight sides

Then area $= \pi r^2 + 2rs$

cost $= 50 = 2s + 32\pi r$

so $s = \dfrac{50-6\pi r}{2} = 25-3\pi r$

substituting back into area equation area $= \pi r^2 + 2r(25-3\pi r)$

$$= \pi r^2 + 50r - 6\pi r^2$$
$$A(r) = -5\pi r^2 + 50r$$

Find maximum by taking derivative, setting it equal to 0

$A'(r) = -10\pi r + 50$

$A'(r) = 0$ when $r = \dfrac{50}{10\pi} = \dfrac{5}{\pi}$

Check that this is a maximum:
$A''(r) = -10\pi$ which is negative, so 2nd deriv. test $\Rightarrow$ maximum

Answer is thus diameter $= 2r = \dfrac{10}{\pi}$ feet

and straight side length $s = 25-3\pi r = 25-3\pi\left(\frac{5}{\pi}\right) = 10$ feet