Solutions to Final for Math Xa

Jan. 17, 1997

1. a) NEI b) $U(t)$ is increasing
   c) $U'(t)$ is decreasing
   d) $U''(t) < 0$
   e) $A''(t) > 0$
   f) NEI

2. $B(t) = B_0\left(1+\frac{K}{100}\right)^{t/5}$

$$2\cancel{B_0} = \cancel{B_0}\left(1+\frac{K}{100}\right)^{t/5}$$

$$\ln 2 = \frac{t}{5}\ln\left(1+\frac{K}{100}\right)$$

$$\frac{t}{5} = \frac{\ln 2}{\ln\left(1+\frac{K}{100}\right)}$$

$$\boxed{t = 5\frac{\ln 2}{\ln\left(1+\frac{K}{100}\right)}}$$

3. a) $r'(x) = f'(x)\cdot g(h(x)) + f(x)\cdot g'(h(x))\cdot h'(x)$

b) $r'(3) = f'(3)\cdot g(h(3)) + f(3)\cdot g'(h(3))\cdot h'(3)$

$$= 2\cdot g(0) + 7\cdot g'(0)\cdot(-2)$$

$$= 2\cdot 5 + 7\cdot\pi(-2)$$

$$\boxed{=10-14\pi}$$

4. a) land allotted to beets =

$$A - \frac{G}{100}A - C - W$$ let's call this B

$\uparrow$ total $-$ (amt used for cattle, corn & wheat)

b) $D =$ amt (\$) for corn $+$ amt (\$) for wheat $+$ amt (\$) for beets

let X = # of dollars he gets for 1 acre of beets

$$D = YC + ZW + X(B)$$

($/acre)(acres) ($/acre)(acres) ($/acre)(# acres of beets)

$$X = \frac{D - YC - ZW}{B}$$

$$\boxed{X = \frac{D - YC - ZW}{A - \frac{G}{100}A - C - W}}$$

5. $f(x) = \frac{1}{2}\ln(3x) = \frac{1}{2}(\ln x + \ln 3)$

x-int. $y = 0$ $0 = \frac{1}{2}\ln(3x)$

$$\ln 3x = 0$$

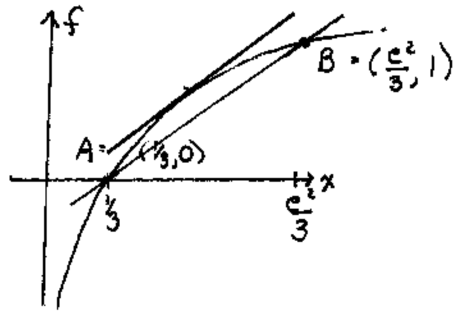
$$3x = 1$$

$$x = \tfrac{1}{3}$$

x-coord $= \frac{e^2}{3} \Rightarrow f\left(\frac{e^2}{3}\right) = \frac{1}{2}\ln\left(\frac{3e^2}{3}\right) =$

$$\frac{1}{2}\ln e^2 = \frac{1}{2}\cdot 2 = 1$$

a)



b) $m = \dfrac{1-0}{\frac{e^2}{3}-\frac{1}{3}} = \dfrac{1}{\frac{e^2-1}{3}} = \dfrac{3}{e^2-1}$

$y - y_1 = m(x - x_1)$ let $(x_1, y_1) = (\frac{1}{3}, 0)$

$$y - 0 = \frac{3}{e^2-1}\left(x - \frac{1}{3}\right)$$

$$\boxed{y = \frac{3}{e^2-1}x - \frac{1}{e^2-1}}$$

c) $f'(x) = \frac{d}{dx}\left[\frac{1}{2}(\ln x + \ln 3)\right] = \frac{1}{2}\left(\frac{1}{x} + 0\right)$

$$= \frac{1}{2x}$$

We want to find x such that

$$\frac{1}{2x} = \frac{3}{e^2-1}$$

$$6x = e^2 - 1$$

$$\boxed{x = \frac{e^2-1}{6} \approx 1.065}$$

d) It isn't necessary to restrict the domain.
Interchange x & y & solve for y

$$x = \frac{1}{2}\ln(3y)$$

$$2x = \ln 3y$$

$$e^{2x} = 3y$$

$$y = \frac{1}{3}e^{2x}$$

$$f^{-1}(x) = \frac{1}{3}e^{2x}$$

6. a) this looks like an exponential, flipped & shifted up.

$y = Ca^x + 7$ when $x = 0$, $y = 4$

$$4 = Ca^0 + 7$$

$$-3 = C$$

$y = -3a^x + 7$ when $x = \frac{2}{3}$, $y = 5$

$$5 = -3a^{2/3} + 7$$

$$-2 = -3a^{2/3}$$

$$\frac{2}{3} = a^{2/3} \rightarrow a = \left(\tfrac{2}{3}\right)^{3/2}$$

so

$$\boxed{y = -3\left(\tfrac{2}{3}\right)^{\frac{3}{2}x} + 7}$$

6b. $\boxed{y = \dfrac{5(x+4)(x)(x-3)}{(x+3)^2\,(x-2)}}$

7. a. $y = (x + x^{-1})^5 \cdot e^{-x}$

so $y' = 5(x+x^{-1})^4(1-x^{-2})\cdot e^{-x} + -e^{-x}(x+x^{-1})^5$

$\boxed{y' = e^{-x}\left(5\left(x+\frac{1}{x}\right)^4\left(1-\frac{1}{x^2}\right) - \left(x+\frac{1}{x}\right)^5\right)}$

b) $y = \ln\left(\dfrac{\sqrt{x}}{x^2 3^x}\right) = \ln\sqrt{x} - \ln(x^2 3^x)$

$y = \frac{1}{2}\ln x - 2\ln x - x\ln 3$

so $y' = \frac{1}{2}\frac{1}{x} - 2\cdot\frac{1}{x} - \ln 3$

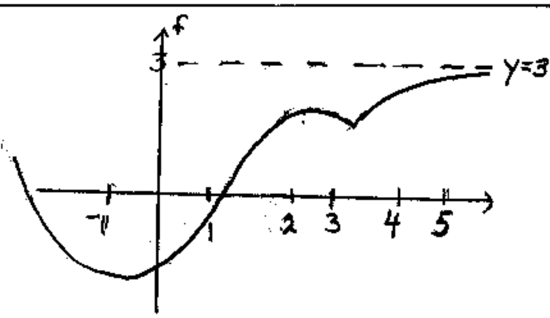
$\boxed{y' = \frac{1}{2x} - \frac{2}{x} - \ln 3}$

c) $y = \pi\left(1 + \frac{\pi}{2}\right)^{5x}$ of the form $b^{xg(x)}$

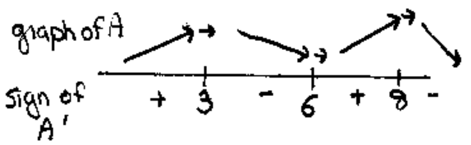
$\boxed{y' = \pi\ \ln\left(1+\frac{\pi}{2}\right)\cdot\left(1+\frac{\pi}{2}\right)^{5x}\cdot 5}$

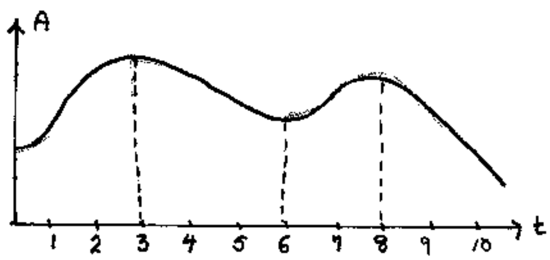
since $\frac{d}{dx}\, b^{g(x)} = \ln b \cdot b^{g(x)} \cdot g'(x)$

8.



9. a) $R(t) = A'(t)$: Key rel'nship

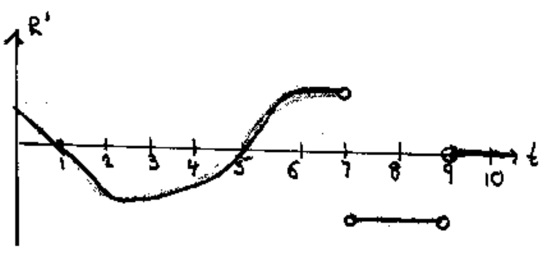




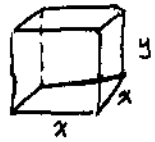
x-coords of extrema: $x=3,\ x=6,\ x=8$

x-coords of pts. of inflection: $x=1,\ x=5,\ x=7$

b)



10)



Cost = cost of base + cost of sides

$105 = 5\cdot(x\cdot x) + 3\cdot 4\cdot xy$

Goal: Maximize volume

$V = x^2 y$

use cost to express y in terms of x

$105 = 5x^2 + 12xy$

$x12y = 105 - 5x^2$

$y = \dfrac{105-5x^2}{12x}$

$V = x^2 y$

$V = x^2\left(\dfrac{105-5x^2}{12x}\right)$

$V = \dfrac{x}{12}(105-5x^2) = \dfrac{105}{12}x - \dfrac{5x^3}{12} \quad x>0$

Find critical points of V.

$V'(x) = \frac{105}{12} - \frac{15}{12}x^2 = 0$

$105 - 15x^2 = 0$

$x^2 = \frac{105}{15} = 7$

$x = \sqrt{7}$

This is our only critical point.

2nd deriv. test: $V''(x) = -\frac{30}{12}x$

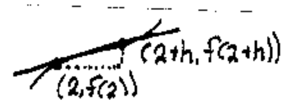
$V''(\sqrt{7}) < 0 \Rightarrow \sqrt{7}$ is a local max, but it's the only critical pt – so there's an absolute max at $x = \sqrt{7} \approx 2.65$

When $x = \sqrt{7}$, $y = \dfrac{105-5\cdot 7}{12\cdot\sqrt{7}} = \dfrac{105-35}{12\sqrt{7}} = \dfrac{70}{12\sqrt{7}} \approx 15.43$

The box of greatest volume is

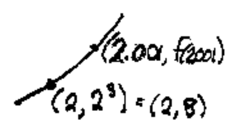
$\sqrt{7} \times \sqrt{7} \times \frac{70}{12\sqrt{7}}$

11. a) $f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$



b) c, d

12. Pick a pt. near $(2, f(2))$ – say $(2.001, f(2.001))$



$f'(2) \approx \dfrac{f(2.001) - f(2)}{.001}$

$= \dfrac{(2.001)^{2.001+1} - 2^3}{.001}$

≈ 17.56

or, even better $f'(2) \approx \dfrac{f(2.0001) - f(2)}{.0001} \approx 17.54$

Note: don't round off numerator before dividing by .001 or .0001.