

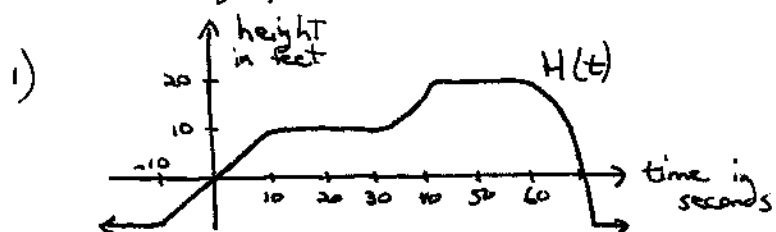
Getting Ready for the Final Part One!

Math X
Jan 100

We'll split up the material over 3 review sessions:

- Topics: ~~#~~
- A. Words to graphs; words to math
 - B. Definition of derivative
 - C. Going from $f \rightarrow f'$; $f' \rightarrow f$; etc.
 - D. Limits - analytically and graphically
 - E. Functions: Polynomials
Exponentials
Logarithms
 - F. Extrema - critical points; local/abs max/min
 - G. Flips, shifts, stretches of functions
 - H. Inverse Functions (graphs and word explanations)
 - I. Differentiation Part 2 - differentiating basic functions
- Product / Chain Rule
 - J. Solving Equations using logs, exponentials
 - K. Optimization problems

① Words to graphs; etc.



- Suppose $t=0$ represents 5 pm today and $H(t)$ gives the height of an object above ground level (positive = above ground)
- a) When is the object's velocity equal to 0?
 - b) When is the object's speed maximum?
 - c) How far away from the ground does the object get?
 - d) What might the object be?

- 2) Suppose $I(t)$ gives the amount of ink left in my pen at time t , where $t=0$ represents noon, and t is measured in minutes, and $I(t)$ is measured in milliliters

What do each of the following mean?

- a) $I(-30) = 5$
- b) $I(0) = 5$
- c) $I(20) = 3$
- d) $|I'|$ is maximal at $t = 15$
- e) $I'(35) = 0$
- f) $I(70)$ is an absolute minimum for $I(t)$
- g) $I(100) = 0$

How would you write an equation to show

h) I have no ink left after one o'clock

i) The pen holds 10 milliliters of ink, and it was full before 11 am

ⓑ Definition of the Derivative

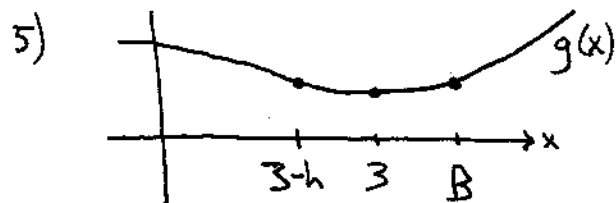
3) Using the usual definition $f(x) = \lim_{h \rightarrow 0} \left(\right)$

find $f'(x)$ if $f(x) = \frac{10}{x}$

b) find $f'(3)$

c) find $f'(0)$

4) Find $g'(x)$ if $g(x) = 3x^2 - 2$ (using the limit definition again!)



Which of the following equals $g'(3)$?

a) $\lim_{B \rightarrow 0} \frac{g(B) - g(3)}{B}$

e) $\lim_{h \rightarrow 0} \frac{g(3-h) - g(3)}{h}$

b) $\lim_{h \rightarrow 3} \frac{g(3) - g(h)}{h}$

f) $\lim_{B \rightarrow 3} \frac{g(B) - g(3)}{B-3}$

c) $\lim_{B \rightarrow 3} \frac{g(3+B) - g(3)}{3+B}$

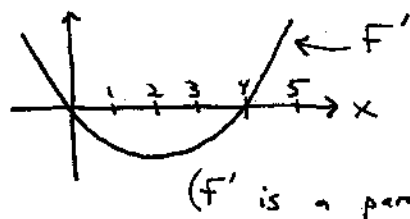
g) $\lim_{B \rightarrow 3} \frac{g(B-3) - g(B)}{3}$

d) $\lim_{h \rightarrow 0} \frac{g(3-h) - g(3)}{-h}$

h) 0

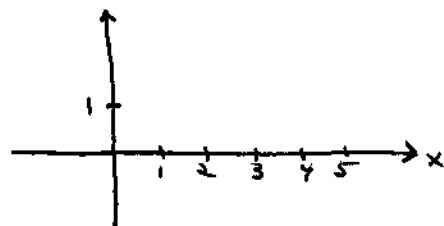
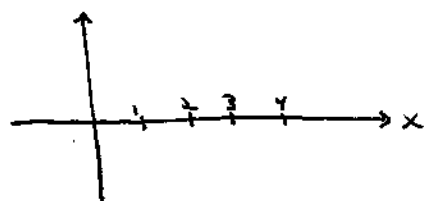
⑥ Going from $f \rightarrow f'$, etc.

a) Given

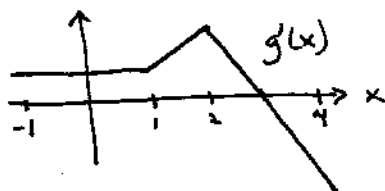


draw a
possible
graph of f
if $f(0) = 1$

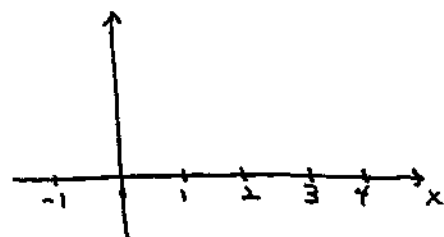
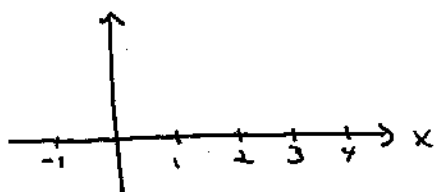
graph f''



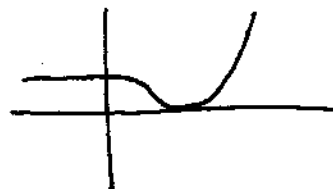
7) Given



graph a
possible $g(x)$
if $g(0)=0$

graph $g''(x)$ 

8) Graph f and f'' if you're given f' :
again let $f(0)=0$



① Limits

9) What is $\lim_{x \rightarrow 0} (x^3 + 5x - 10)$?

10) If $f(x) = 3x^3 - 2x$ (a) what is $\lim_{x \rightarrow \infty} f(x)$?

(b) what is $\lim_{x \rightarrow 0} f'(x)$?

(c) what is $\lim_{x \rightarrow -\infty} f''(x)$?

11) If $H(x) = \left(\frac{4}{x^2} + \frac{3}{x-3} + 1999 \right)$

(a) then what is $\lim_{x \rightarrow 0} H(x)$?

(b) what is $\lim_{x \rightarrow 2} H(x)$?

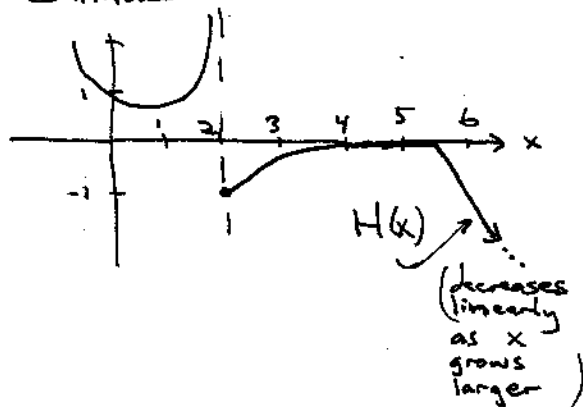
(c) $\lim_{x \rightarrow \infty} H(x)$?

(d) $\lim_{x \rightarrow -\infty} H(x)$?

12) What is a) $\lim_{x \rightarrow -\infty} 1999x^{1998}$? b) $\lim_{x \rightarrow -\infty} 1998x^{1999}$?

Limits continued

13)



(a) What is $\lim_{x \rightarrow 2^+} H(x)$?

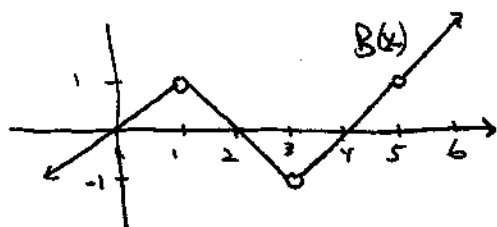
(b) What is $\lim_{x \rightarrow 2} H(x)$?

(c) What is $\lim_{x \rightarrow \infty} H(x)$?

(d) What is $\lim_{x \rightarrow 5^+} H(x)$?

(e) What is $\lim_{x \rightarrow 5} H'(x)$?

14)



(a) What is $\lim_{x \rightarrow 1} B(x)$?

(b) What is $\lim_{x \rightarrow 2} B'(x)$?

(c) What is $\lim_{x \rightarrow \infty} B'(x)$?

(d) What is $\lim_{x \rightarrow 5} B(x)$?

(e) What is $\lim_{x \rightarrow 1} B'(x)$?

(f) What is $\lim_{x \rightarrow 5} B'(x)$?

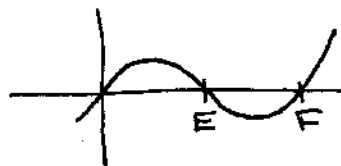
⑤ Functions - Polynomials

Remember if you can factor it then you should probably go ahead (if it's easy)

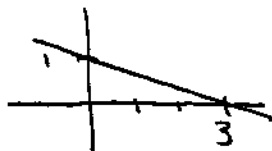
15) What are the roots of $(x-3)^2(x^2-25)$?

16) Sketch a graph of $x(x-2)(x+2)$

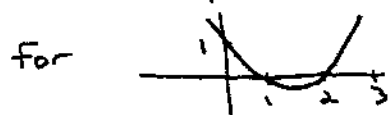
17) What is a possible function for



18) Give the function for



19) Give a possible function



20) IF $B(x) = 3x^5 + 2x^3 - p \cdot x + 27$

a) What is $\lim_{x \rightarrow \infty} B(x)$?

b) $\lim_{x \rightarrow -\infty} B(x)$

c) What is the slope of $B(x)$ at $x=0$?

d) What is $B(x)$'s y-intercept

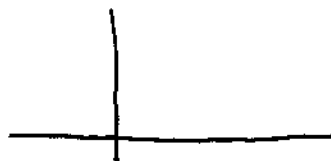
e) What is the maximum # of roots of $B(x)$?

f) What is the maximum # of turning pts of $B(x)$?

g) Does $B(x)$ have an absolute max. or min.?

Exponential Functions

Know basic shape 2) graph $3^x + 2$



2) If the number of red pens, $R(t)$, is 24 at time $t=0$ (t measured in weeks) and pens decrease at a rate of 25% every two weeks, give a formula for $R(t)$.

Final Push for the Final Part Two

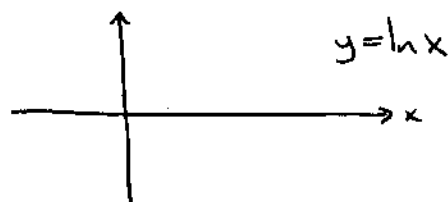
Math Xa

continuing the series of topics:

- E. Functions - Logarithms
- F. Extrema - critical pts, abs/loc max min
- G. Flips, shifts, stretches of functions
- H. Inverse functions

E. Functions - Logarithms

know shape of logarithm graph:



and basic properties:

1) $a^b = x$ is the same as $\log_a x = b$

2) $\ln(a \cdot b) =$

3) $\ln\left(\frac{a}{b}\right) =$

4) $\ln[w^2] =$

5) Simplify the following:

a) $2 \log x - \log(3x) + 2 \log(\sqrt{3})$

b) $\log_3 27 + \frac{1}{2} \log_{12} 144 - \log(1,000,000)$

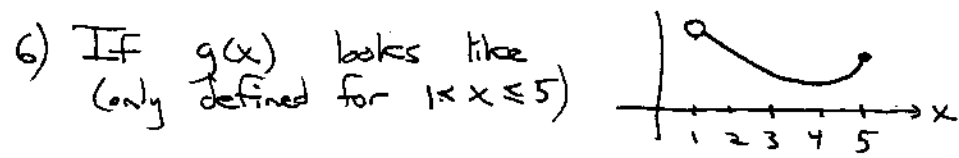
c) $\ln(e^x) - 3 \log_5(25) + \log_{100} 10$

d) $\log 15 - \log 3 - \log(\ln(e^5))$

Ⓕ Extrema!

know Definition of Critical point:

Critical Points are the values which might be Extrema
Defn of Extrema:



What are $g(x)$'s Extrema?

7) How about $g(x) = x^3 - x$? What are its Extrema,
and what are its critical points?

8) Use the second derivative test to check if
 $h(x) = \frac{1}{3}x^3 + x^2 + x$ has a local max or min.
(first find any critical points!)

9) What does the first derivative test say about the
extrema for $B(x) = x^2 - 6x + 9$?

⑥ Flips, Stretches, etc.

Given $y = f(x)$ what happens to the graph if...

10) $y = 3f(x)$

11) $y = \frac{1}{2}f(x)$

12) $y = -f(x)$

13) $y = f(x) + 5$

14) $y = f(x - 2)$

15) $y = f(x + 3)$

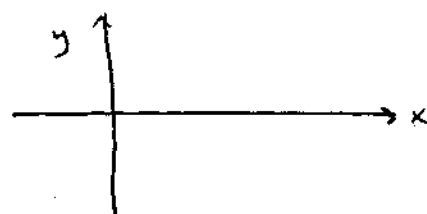
16) $y = f(-x)$

17) $y = f(5x)$

18) Graph a) $y = (x - 3)^2 + 2$



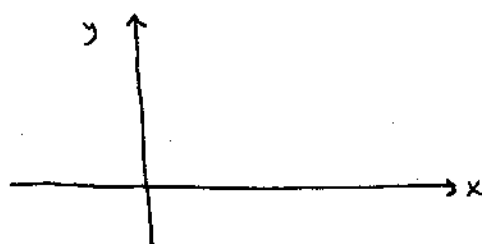
b) $y = -\ln(x^2)$



c) $y = 3e^{-x} - 1$



d) $y = \frac{1}{2}(2^x) + \frac{1}{2}$



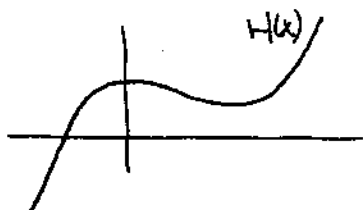
④ Inverse Functions

know the trick for finding inverses algebraically

19) If $f(x) = \frac{x+2}{x-1}$ then what is $f^{-1}(x)$?

20) Where is $f(x)$ defined (what is its domain)?

21) Is $H(x)$ invertible?



22) I can make $B(t)$ brownies in t minutes. On the day before the last class I made D brownies in C minutes.

Describe in words each of the following:

a) $B(30) = 65$

b) $B^{-1}(65) = 30$

c) $B^{-1}(230 + D)$

d) $B^{-1}(D - 25)$

e) $B(C + 10)$

f) $B(C) + 10$

g) $3(B^{-1}(D))$

Getting Ready for the Final - Part 3!!

Math Xa

Remaining topics include:

- I. Differentiation - basic functions and Product/Quotient/Chain Rules
- J. Solving Equations - using logs/exp.s
- K. Optimization Problems

I. Differentiation

Know the derivatives of all the basic functions (polynomials, powers of x like $\frac{1}{x^2}$, exp.s, logs)

Differentiate each of the following functions

1) $7x^4 - 2x + \frac{1}{x}$

2) $ax^3 + be^x$

3) $12e^x + \log_{12} x$

4) $5 \ln(6x^5)$

5) $\frac{1}{w \cdot x^2} - \frac{1}{z \cdot (x^{-1})}$

6) $2^{(4x)} - 4^x$

Recall the Product Rule:

the derivative of $f(x) \cdot g(x)$ is ?...

Find the derivative for each of the following
(I didn't know it was lost!)

7) $(2x+3) \cdot (e^x)$

8) $(\ln(x^5)) (5^x + 3x)$

9) $(12x^5 + 5)(x^3 - 1)$

Quotient Rule: recall that the derivative of $\frac{f(x)}{g(x)}$ is ?...

Find the derivative of
10) $\frac{3x^2}{5x+2}$

11) $\frac{2^x}{3^x + 5^x}$

12) $\frac{\ln(x^4)}{\log_{12}(x^4)}$

Finally, Chain Rule

recall that the derivative
of $f(g(x))$ (i.e. a composition of
functions $f(x)$ and $g(x)$)

is ?...

So find the derivatives for each of the following composition functions

13) $(3x+12)^5$

14) $(\ln(x)+1)^{98}$

15) e^{kx}

16) $\frac{1}{e^{\log_5 x}}$

17) $H(14x+2)$

18) $H(5x^3 + e^x)$

⑤ Solving Equations using Logs and Exponential Functions

- pull down powers involving x by taking logs
- undo logarithms by taking exponents

Solve each of the following for x (solve exactly, not just numeric approximations)

19) $5^{2x+3} = 27$

20) $\ln x = 3$

21) $8^x = 3 \cdot 12^x$

$$22) \log_{15} x - \log_{15} w = \log_{15} z$$

$$23) e^{3x} \cdot e^{-2x} = e^{\log_5 19}$$

$$24) \log_{12} x^2 + \log_{12} x^3 = 5$$

(K) Optimization Problems

- 1) Name variables you need (width = w, radius = r, etc.)
- 2) Find equations:
 - a) linking variables if you've got more than one
 - b) giving the function you want to optimize
- 3) Usual trick $\Rightarrow$ solve for one of the variables and substitute the result into the function you want to optimize
- 4) Optimize! (usually just find derivative, set = 0, check max/min)

- 25) A family goes on vacation, and they decide to drive. Assuming that they need to drive 500 miles and their car uses up $.002s^2 - .005s + 2.5$ gallons per hour, where s is the speed of the car, then if they drive at a constant speed, then what should that speed be to minimize the amount of gas that they use?

Answers to Math X Review Sheets

Part One

- 1) a) velocity = 0 when graph is horizontal, so $t < -10$ and $10 < t < 30$
b) speed = absolute value of velocity
(i.e. speed is always positive, or 0). Speed is maximum right after $t = 70$
c) 20 feet
d) an elevator, perhaps, stopping every 10 feet to let people on, and then dropping a little too quickly back to the basement at the end!
- 2) a) 5 milliliters of ink left in pen at 30 minutes before noon (11:30 am)
b) There are 5 milliliters of ink in the pen at noon ($t = 0$)
c) At 12:20 there are 3 milliliters of ink in the pen
d) The rate of flow of the ink out of the pen is fastest at 12:15 pm
e) No ink is being used right at 12:35
f) No more ink is used after 1:10 ($t = 70$)
g) All the ink is used up by 1:40 ($t = 100$)
h) $I(x) = 0$ for $x > 60$
i) $I(x) = 10$ for $x < -60$

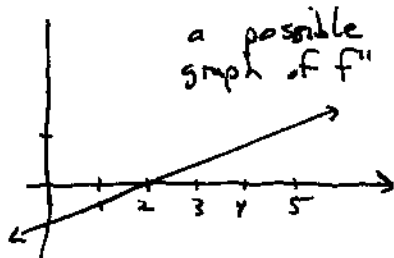
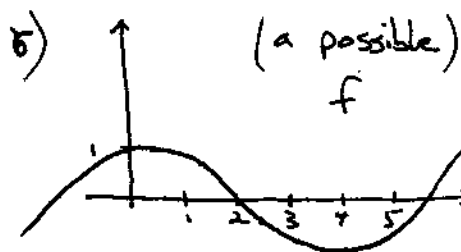
3) $f(x) = \frac{10}{x}$, so $f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{\frac{10}{x+h} - \frac{10}{x}}{h} \right)$
a)
$$= \lim_{h \rightarrow 0} \left(\frac{\frac{x \cdot 10}{x(x+h)} - \frac{(x+h) \cdot 10}{(x+h)x}}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{\frac{10x - 10x - 10h}{x(x+h)}}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{-10h}{x(x+h)h} \right)$$
$$= \lim_{h \rightarrow 0} \left(\frac{-10}{x(x+h)} \right) = \frac{-10}{x^2}$$

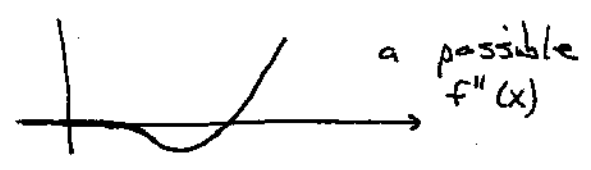
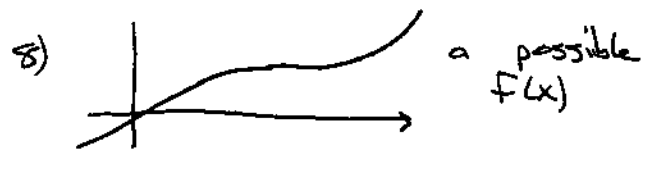
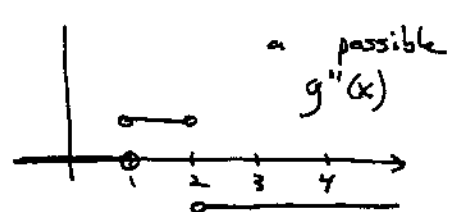
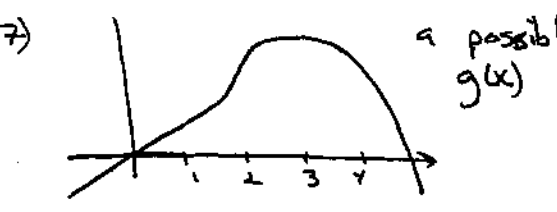
b) $f'(3) = \frac{-10}{3^2} = -\frac{10}{9}$

c) $f'(0)$ wow, it's undefined (trick question!)

4) $g(x) = 3x^2 - 2$ so $g'(x) = \lim_{h \rightarrow 0} \left(\frac{g(x+h) - g(x)}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{3(x+h)^2 - 2 - (3x^2 - 2)}{h} \right)$
$$= \lim_{h \rightarrow 0} \left(\frac{3(x^2 + 2hx + h^2) - 2 - 3x^2 + 2}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{3x^2 + 6hx + 3h^2 - 3x^2}{h} \right)$$
$$= \lim_{h \rightarrow 0} \left(\frac{6hx + 3h^2}{h} \right) = \lim_{h \rightarrow 0} (6x + 3h) = 6x$$

5) Just d, f and h





9) -10

10) a) $+\infty$ b) $f'(x) = 9x^2 - 2$, $\lim_{x \rightarrow 0} f'(x) = -2$ c) $f''(x) = 18x$, $\lim_{x \rightarrow -\infty} f''(x) = -\infty$

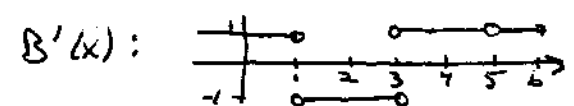
11) a) $+\infty$ (graph the function to check) c) 1999

b) $\frac{4}{2^2} + \frac{3}{2-3} + 1999 = 1 - 3 + 1999 = 1997$ d) 1999

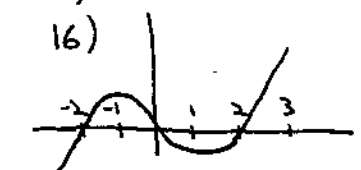
12) a) even exponent, positive coefficient $\Rightarrow \lim_{x \rightarrow -\infty} = +\infty$ b) odd degree, positive $\Rightarrow \lim_{x \rightarrow -\infty} = -\infty$

13) a) -1 b) undefined c) $-\infty$ d) 0 e) 0

14) a) 1 b) -1 c) 1 d) 1 e) undefined f) 1



15) roots at $x=3$, $x=\pm 5$

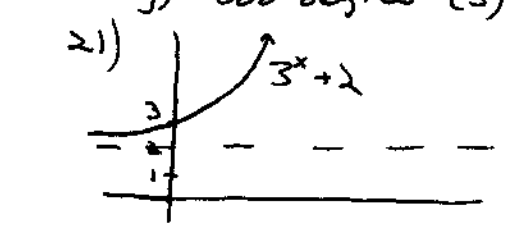


17) $x(x-E)(x-F)$

18) slope $= -\frac{1}{3}$, y-intercept $= 1$, so $y = -\frac{1}{3}x + 1$

19) $y = k(x-1)(x-2)$ and $y = 1$ when $x = 0$, so $1 = k(-1)(-2)$
means $k = \frac{1}{2}$, so $y = \frac{1}{2}(x-1)(x-2)$

- 20) a) $+\infty$
b) $-\infty$
c) $B'(x) = 15x^4 + 6x^2 - p$, so $B'(0) = \text{slope at } x=0 = -p$
d) 27
e) degree 5 $\Rightarrow$ max. of 5 roots
f) maximum of 4 turning points
g) odd degree (5), no absolute max or min



22) $R(t) = 24 (.75)^{t/2}$

Answers for Part 2

1) $\log_a x = b$ 2) $\ln a + \ln b$ 3) $\ln a - \ln b$ 4) $2 \cdot \ln(w)$

5) a) $2 \log x - (\log 3 + \log x) + \log (\sqrt{3})^2 = 2 \log x - \log 3 - \log x + \log 3 = \log x$

b) $3 + \frac{1}{2}(2) - \log_{10}(10^6) = 3 + 1 - 6 = -2$

c) $x - 3(2) + \frac{1}{2} = x - 5\frac{1}{2}$

d) $\log 5 + \log 3 - \log 3 - \log 5 = 0$

6) local, absolute min. at $x=4$, no local or abs. max's

7) $g'(x) = 3x^2 - 1$, so $g' = 0$ when $x = \pm \frac{1}{\sqrt{3}}$ are the critical points.

$g''(x) = 6x$, and $g''(-\frac{1}{\sqrt{3}}) < 0 \Rightarrow x = -\frac{1}{\sqrt{3}}$ is a maximum

$g''(+\frac{1}{\sqrt{3}}) > 0 \Rightarrow x = +\frac{1}{\sqrt{3}}$ is a minimum

(2nd derivative test, concave up $\cup \Rightarrow$ minimum, etc)

8) $h'(x) = x^2 + 2x + 1 = (x+1)^2$ so only critical point is $x = -1$

$h''(x) = 2x + 2$, and $h''(-1) = 0 \Rightarrow$ 2nd derivative fails to tell us anything!

to check what's happening look at first derivative again: $h'(x) = (x+1)^2$ is always positive (or 0 at $x = -1$) so there aren't any local mins, maxs.

$x = -1$ is in fact an inflection point, since

$h''(x) = 2x + 2$ changes from negative to pos. at $x = -1$.

9) $B'(x) = 2x - 6$, so critical point at $x = 3$,
here $B'(x)$ goes from negative to positive,
so must be a local/absolute min. at $x = 3$

10) vertical stretch by 3 times

11) vertical shrink (down by $\frac{1}{2}$)

12) flip over x-axis

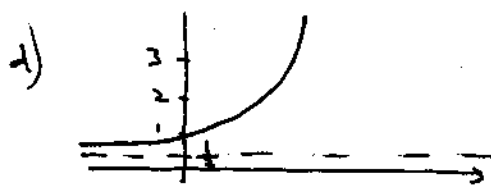
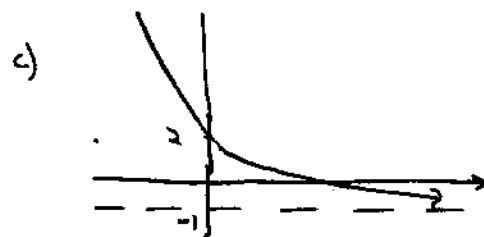
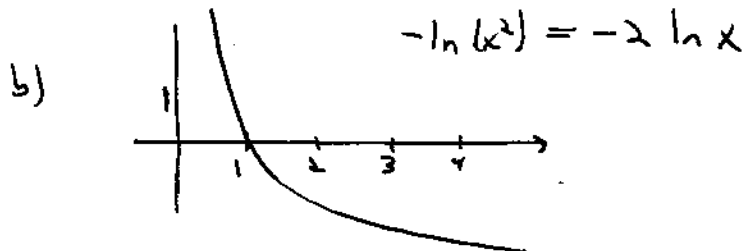
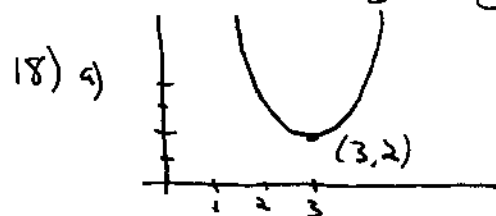
13) shift up by 5

14) shift to right by 2

15) shift to left by 3

16) flip over y-axis

17) horizontal shrink



19) $y = \frac{x+2}{x-1}$, then switch x, y $x = \frac{y+2}{y-1}$, solve for y

20) $x(y-1) = y+2$, so $xy - x = y+2$, $xy - y = x+2$
 $y(x-1) = x+2 \Rightarrow y = \frac{x+2}{x-1}$

21) Not invertible, fails horizontal line test

22) a) I bake 65 brownies in 30 minutes B: minutes $\rightarrow$ brownies

b) It takes 30 minutes to bake 65 brownies B': brownies $\rightarrow$ minutes

c) The time it takes to bake 230 more brownies than I made the day before the last class

d) The time it takes to bake 25 fewer brownies than I made the day before the last class

e) The number of brownies I could bake if I spent 10 more minutes baking brownies than I did the day before the last class

f) $= D + 10$, 10 more brownies than I made the day before the last class

g) $= 3C$, Three times the amount of time I spent baking brownies the day before the last class.

Part Three

1) $28x^3 - 2 - \frac{1}{x^2}$

2) $3ax^2 + 6e^x$

3) $12e^x + \frac{1}{(\ln 12)x}$

4) $= 5(\ln 6 + \ln x^5) = 5\ln 6 + 25\ln x$, derivative $= \frac{25}{x}$

5) $\frac{-2}{w \cdot x^3} - \frac{1}{2}$

6) $\ln(16) \cdot 16^x - \ln(4) \cdot 4^x$

7) $(2x+3)e^x + 2e^x = (2x+5)e^x$

8) $\ln(x^5) \left((\ln 5) 5^x + 3 \right) + \left(\frac{5}{x} \right) (5^x + 3x)$

9) $(12x^5 + 5)(3x^2) + 60x^4(x^3 - 1)$

10) $\frac{(5x+2)(6x) - (3x^2)5}{(5x+2)^2}$

11) $\frac{(3^x + 5^x)(\ln 2) 2^x - 2^x (\ln 3 3^x + \ln 5 5^x)}{(3^x + 5^x)^2}$

12) $= \frac{4 \ln 4}{4 \log_2 x} = \frac{\ln x}{\log_2 x}$, derivative $= \frac{(\log_2 x) \frac{1}{x} - \ln x \left(\frac{1}{(\ln 2)x} \right)}{(\log_2 x)^2}$

13) $5(3x+12)^4 \cdot 3$

14) $\frac{98(\ln x + 1)^{97}}{x}$

15) ke^{kx}

16) $= e^{-\log_5 x}$, derivative is $\left(e^{-\log_5 x} \right) \left(\frac{-1}{(\ln 5)x} \right)$

17) $14 \cdot H'(14x+2)$

18) $(15x^2 + e^x) H'(5x^3 + e^x)$

19) $(2x+3)\ln 5 = \ln 27$, $2x+3 = \frac{\ln 27}{\ln 5}$, $x = \frac{1}{2} \left(\frac{\ln 27}{\ln 5} - 3 \right)$

20) $x = e^3$ 21) $\left(\frac{8}{12} \right)^x = 3$, $x \ln \left(\frac{2}{3} \right) = \ln 3$, $x = \frac{\ln 3}{\ln \left(\frac{2}{3} \right)}$

22) $x = 2 \cdot w$

23) $x = \log_5 19$

24) $x = 0$

25) Minimize $\left(\frac{500}{s} \right) (0.0025s^2 - 0.05s + 2.5) = 5 - 2.5 + \frac{1250}{s}$ Derivative is $1 - \frac{1250}{s^2}$

set equal to zero, find minimum when $s = \sqrt{1250}$ or about 35 mph