

Last Lab of the First Semester
Solutions

Differentiate the following.

The solutions below have not been simplified because we want the use of the Chain, Product and Quotient Rules to be quite transparent.

1. $y = \pi \ln(g(1-x) + g(1+x))$

$$y' = \pi \frac{1}{g(1-x) + g(1+x)} \cdot [g'(1-x) \cdot (-1) + g'(1+x) \cdot (1)]$$

2. $y = f\left(\frac{\sqrt{1-x^2}}{e^x}\right)$ Rewrite: $y = f[(1-x^2)^{1/2} \cdot e^{-x}]$

$$y' = f'\left(\frac{\sqrt{1-x^2}}{e^x}\right) \cdot \left[e^{-x} \frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot (-2x) + e^{-x} \cdot (-1) \cdot (1-x^2)^{1/2}\right]$$

3. $y = \frac{\pi^2}{5^{3x^2+4}}$ Rewrite: $y = \pi^2 5^{-3x^2-4}$

$$y' = \pi^2 \ln 5 \cdot 5^{-3x^2-4} \cdot (-6x)$$

4. $y = f[g(x)h(x)]$

$$y' = f'[g(x)h(x)] \cdot [g'(x)h(x) + g(x)h'(x)]$$

5. $y = \log_5(2^{3^x} + 1)$

$$y' = \frac{1}{\ln 5} \cdot \frac{1}{2^{3^x} + 1} \cdot (\ln 2 \cdot 2^{3^x} \cdot \ln 3 \cdot 3^x)$$

6. $y = \frac{f(x^3)[g(x)]^2}{h(e^{5x})}$

$$y' = \frac{h(e^{5x}) \cdot [f'(x^3) \cdot (3x^2) \cdot [g(x)]^2 + f(x^3) \cdot 2[g(x)] \cdot g'(x)] - f(x^3)[g(x)]^2 \cdot h'(e^{5x}) \cdot e^{5x} \cdot 5}{[h(e^{5x})]^2}$$

Who wrote this problem?!

7. $y = \ln\left(\frac{x^2+2x}{x(x+1)^2}\right)$ Rewrite: $y = \ln(x^2+2x) - \ln x - 2\ln(x+1)$

$$y' = \frac{2x+2}{x^2+2x} - \frac{1}{x} - \frac{2}{x+1}$$

8. $y = f(g(x)) + h(j(x))$

$$y' = f'(g(x)) \cdot g'(x) + h'(j(x)) \cdot j'(x)$$

9. $y = \frac{f(\ln(4x))}{g\left(\frac{1}{1-x}\right)}$ Rewrite: $y = \frac{f(\ln(4x))}{g((1-x)^{-1})}$

$$y' = \frac{[g(1-x)^{-1} \cdot f'(\ln(4x)) \cdot \frac{4}{4x}] - g'(1-x)^{-1} \cdot (-1)(1-x)^{-2} \cdot (-1) \cdot f(\ln(4x))}{[g(1-x)^{-1}]^2}$$

10. $y = \frac{(\pi e^2 + x \ln 3)(2x - .5[f(x)]^2)}{\sqrt{2x^5 - 4x}}$

It might be a tad easier to multiply out the numerator first - but not much easier. The answer is more than one line- so I've split the quotient rule up at the minus sign.

$$y' = \frac{\sqrt{2x^5 - 4x} \cdot [\ln 3 \cdot (2x - .5[f(x)]^2) + (\pi e^2 + x \ln 3)(2 - f(x) \cdot f'(x))]}{2x^5 - 4x}$$

$$\frac{(\pi e^2 + x \ln 3)(2x - .5[f(x)]^2) \cdot \frac{1}{2} \cdot (2x^5 - 4x)^{-1/2} \cdot (x^4 - 4)}{2x^5 - 4x}$$

A telephone cable is to be laid under a river from point A to point C, and then underground to point B. The river is 20 m wide and the distance from point B to the point directly across the river from A is 40 m.

- If it costs 1.5 times as much to lay the cable under the river as it does to put it underground, find where point C should be located to minimize the cost. (from Pasachof)
- Does your answer change if it costs twice as much under water?
- Does your answer change if it costs the same under water as under land?

Let x = the distance (in meters) from point C to the point directly across the river from A.
Then the distance between points C and B is $40 - x$.

Total Cost = Cost of cable under water + Cost of cable under land

Cost of cable = (cost per meter)(number of meters)

The number of meters of cable under water is given by $\sqrt{400 + x^2}$.

The number of meters of cable under land is given by $40 - x$.

Notice that $x \in [0, 40]$, where $x = 0$ corresponds to laying the least cable possible under water and $x = 40$ corresponds to laying all the cable under water.

- Suppose that the cost of laying cable under land is k dollars per meter. Then the cost of laying cable under the river is $1.5k$ dollars per meter.

$$\text{Total Cost} = 1.5k\sqrt{400 + x^2} + k(40 - x).$$

But k is a positive constant, so it only vertically stretches or shrinks the total cost function; it doesn't change the x values of the extrema. So let's get rid of the k and let C be a function proportional to the cost.

$$C = 1.5\sqrt{400 + x^2} + (40 - x)$$

where $x \in [0, 40]$.

Goal: Minimize $C(x)$.

First let's find the critical points.

$$C'(x) = 1.5(1/2)(400 + x^2)^{-1/2} \cdot (2x) - 1$$

$$C'(x) = 1.5 \frac{x}{\sqrt{400 + x^2}} - 1$$

C' is never undefined. Set C' equal to zero.

$$1.5 \frac{x}{\sqrt{400 + x^2}} = 1$$

$$1.5x = \sqrt{400 + x^2}$$

$$(1.5)^2 x^2 = 400 + x^2$$

$$((1.5)^2 - 1) x^2 = 400$$

$$x^2 = \frac{400}{(1.5)^2 - 1}$$

$$x = \frac{20}{\sqrt{(1.5)^2 - 1}}$$

(Note that x must be non-negative.)

So the critical points are:

$$x = 0, x = 40, \text{ and } x = \frac{20}{\sqrt{(1.5)^2 - 1}} = \frac{40}{\sqrt{5}} \approx 17.89$$

Options: The minimum must occur at one of these three points. We could evaluate C at each of these points and see which is smaller. Or we could use the first or second derivative test for $x = \frac{40}{\sqrt{5}}$ and if it is a local minimum then it must be the minimum on this interval.

Choose one line of reasoning and follow it through. You will find that the minimum is at $x = \frac{40}{\sqrt{5}}$.

- b) For this part of the problem the role of 1.5 is replaced by 2. The critical points turn out to be $x = 0$, $x = 40$, and $x = \frac{20}{\sqrt{3}}$. Again, there are three approaches to identifying the minimum. Whichever you choose you'll see that $x = \frac{20}{\sqrt{3}}$ gives the minimum.
- c) Here you will not find a place where the derivative is zero. The only critical points are the endpoints. The cable should be laid under the river - in a straight line from point A to point B.