Appendix B: Expanding Algebraic Expressions

Algebraic expressions often involve parentheses (i.e. brackets). For example,

$x(1 + x)$ or $(x + 3)(x + 7)$.

This appendix concentrates on multiplying out algebraic expressions that involve parentheses. This operation is often described as expanding the brackets.

The Distributive Property

The distributive property of real numbers can be expressed through a pair of equations:

$a \cdot (b + c) = a \cdot b + a \cdot c$

$(a + b) \cdot c = a \cdot c + b \cdot c$,

where a, b and c are real numbers. The distributive property tells you how to multiply out algebraic expressions involving parentheses.

Example B.1

Expand the following expressions and simplify as much as you can. (Use the Laws of Exponents to simplify.)

a) $x \cdot (1 + x)$.

b) $(2^x + x) \cdot x^2$.

c) $\sqrt{x} \cdot (x^2 + 1)$

Solution:

a) $x \cdot (1 + x) = (x \cdot 1) + (x \cdot x) = x + x^2$.

b) $(2^x + x) \cdot x^2 = (2^x \cdot x^2) + (x \cdot x^2) = 2^x \cdot x^2 + x^3$.

c) $\sqrt{x} \cdot (x^2 + 1) = \sqrt{x} \cdot x^2 + \sqrt{x} \cdot 1 = x^{1/2} \cdot x^2 + \sqrt{x} = x^{5/2} + \sqrt{x}$. (See Appendix E for adding fractions.)

Example B.2

Expand the following expressions and simplify as much as you can.

a) $(x + 3)^2$.

b) $(x - 3)^2$.

c) $(x + 3) \cdot (x - 3)$.

Solution:

a) $(x + 3)^2 = (x + 3)(x + 3) = x\cdot(x + 3) + 3\cdot(x + 3) = (x\cdot x) + (x\cdot 3) + (3\cdot x) + (3\cdot 3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$.

b) $(x - 3)^2 = (x - 3)(x - 3) = x\cdot(x - 3) - 3\cdot(x - 3) = (x\cdot x) - (x\cdot 3) - (3\cdot x) + (3\cdot 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$.

c) $(x + 3)(x - 3) = x\cdot(x - 3) + 3\cdot(x - 3) = (x\cdot x) - (x\cdot 3) + (3\cdot x) - (3\cdot 3) = x^2 - 3x + 3x - 9 = x^2 - 9$.

The three algebraic expressions from Example B.2 illustrate three important special cases of expanding brackets. These three special cases will be used to help with factoring algebraic expressions in Appendix C. In general, for real numbers a and b:

$(a + b)^2 = a^2 + 2ab + b^2$. A "perfect square."

$(a - b)^2 = a^2 - 2ab + b^2$. A "perfect square."

$(a + b)(a - b) = a^2 - b^2$. The "difference of two squares."

When each factor includes two terms, the product normally contains four terms:

$(a + b)(c + d) = ac + ad + bc + bd$,

although as Example B.2 illustrates, sometimes "like" terms can be combined.

Example B.3

Expand the following expressions and simplify as much as you can. If you can combine like terms, indicate which terms you can combine.

a) $(x + 3)(x + 7)$.

b) $(3^x + 7)(\frac{1}{3^x} + 2)$.

c) $(4 - 3x)^2$.

Solution:

a) $(x + 3)(x + 7) = x^2 + 3x + 7x + 21 = x^2 + 10x + 21$. The two terms "$3x$" and "$7x$" have the same power of x and may be combined.

b) $(3^x + 7)(\frac{1}{3^x} + 2) = \frac{3^x}{3^x} + \frac{7}{3^x} + 3^x\cdot 2 + 14 = 1 + \frac{7}{3^x} + 3^x\cdot 2 + 14 = \frac{7}{3^x} + 3^x\cdot 2 + 15$. The two constant terms may be combined. (See Appendix E for more information on multiplying fractions.)

c) $(4 - 3x)^2 = (4 - 3x)(4 - 3x) = 16 - 12x - 12x + 9x^2 = 16 - 24x + 9x^2$. The two terms "$-12x$" have like powers of x and may be combined.

Exercises for Appendix B

For Problems 1-5, simplify the quantity as much as possible.

1. $-(t + 2) + 3(1 - 2t)$

2. $(y + 9) - 2(2 - \mathrm{y})$.

3. $7(h + 1) + (7h + 1)$.

4. $(x + 3)6 + 3(x + 6)$.

5. $-s + 1 + -1(s + 1)$.

For Problems 6-15, expand each of the following expression as much as possible.

6. $x(x^2 + 1)$.

7. $9y^2(2y + 3)$.

8. $4(r + 4)^2 + 1$.

9. $(2w - 1)(4p + 1)$.

10. $(2w - 1)(2w - 1)$.

11. $6y(x - y) + 3(y + 1)^2$.

12. $9w(w - w^2) - 9w^2$.

13. $(z + 2)(y + 3)$.

14. $-6(x + 1)^2$.

15. $(5u - 1)(5u + 1)$

For Problems 16-20, simplify by expanding and collecting like terms.

16. $(t^2 + 1)(t + 1) - t^3$.

17. $(r + 1)\left(\frac{r^2}{2} + 1\right)$.

18. $(x^2 - y^2)(x + y)$.

19. $(x^2 - y^2)^2$.

20. $\left(\frac{e^u + e^{-u}}{2}\right)^2 - \left(\frac{e^u - e^{-u}}{2}\right)^2$.

Answers to Exercises for Appendix B

1. $-7t + 1$.
2. $5y + 5$.
3. $14h + 8$.
4. $9x + 36$.
5. $-2s$.
6. $x^3 + x$.
7. $18y^3 + 27y^2$.
8. $4r^2 + 32r + 65$.
9. $8wp + 2w - 4p - 1$.
10. $4w^2 - 4w + 1$.
11. $6xy - 3y^2 + 6y + 3$.
12. $-9w^3$.
13. $zy + 3z + 2y + 6$.
14. $-6x^2 - 12x - 6$.
15. $25u^2 + 1$.
16. $t^2 + t + 1$.
17. $(r^3 + r^2)/2 + r + 1$.
18. $x^3 + x^2y - xy^2 - y^3$.
19. $x^4 - 2x^2y^2 + y^4$.
20. 1.