Appendix H: Interpreting and Working with Inequalities

The goal of interpreting inequalities

The goal of solving an equation was usually to find one or more numbers that could be plugged into the equation. The goal of interpreting an inequality is to determine a range of numerical values that work in the inequality.

Example H.1

Use words to describe the range of numerical values that work in each of the following inequalities.

a) $-2 \leq x \leq 7$.

b) $-2 < x < 7$.

c) $100 > r \geq 4$.

d) $-\infty < y < 9$.

Solution:

a) The range of numerical values is all real numbers between −2 and 7, including both −2 and 7.

b) The range of numerical values is all real numbers between −2 and 7, but not including either −2 or 7.

c) The range of numerical values is all real numbers between 4 and 100, including 4, but not including 100.

d) The range of numerical values is all real numbers less than 9, not including 9.

Manipulating inequalities

Inequalities may be manipulated in the many of the same ways as equations can be manipulated. The main difference between manipulation of inequalities and manipulation of equations lies in the effects of the distributive law.

Suppose that a, b and c are all real numbers, and that $a < b$. Then:

$a + c < b + c$

$a - c < b - c$

$c \cdot a < c \cdot b$, provided that c is a positive number.

$c \cdot a > c \cdot b$, provided that c is a negative number.

The last two rules indicate how manipulating inequalities differs from manipulating equation. If you multiply (or divide) an inequality by a negative number, then this reverses the direction of the inequality.

Working with linear inequalities

Linear inequalities are like linear equations, except that the equality (=) is replaced by an inequality ($<$, $>$, $\leq$, or $\geq$). Linear inequalities may be manipulated in exactly the same way as linear equations, except that multiplying (or dividing) by a negative number reverses the direction of the inequality.

Example H.2

Use words to describe the range of numerical values that work in each of the following inequalities.

a) $-2x + 9 \leq x - 7$.

b) $\dfrac{x+1}{-2} \geq 4$.

c) $2 < 3x + 1 < 11$.

Solution:

a) If you were solving the linear equation

$-2x + 9 = x - 7$,

a reasonable strategy would be to group like terms, and then make x the subject of the equation. The working to implement this strategy would look something like this:

$-2x - x = -7 - 9$	Group like terms.
$-3x = -16$	Simplify by adding the like terms together.
$x = \dfrac{-16}{-3} = \dfrac{16}{3}$.	Make x the subject by dividing both sides by -3.

The working to interpret the inequality is similar, except that you have to remember that multiplying (or dividing) by a negative number reverses the direction of the inequality.

$-2x - x \leq -7 - 9$	Group like terms.
$-3x \leq -16$	Simplify by adding the like terms together.
$x \geq \dfrac{-16}{-3} = \dfrac{16}{3}$.	Make x the subject by dividing both sides by -3. The direction of the inequality is reversed.

The range of numerical values that satisfy the inequality is the set of numbers that are greater than or equal to 16/3.

b) Multiplying both sides of the inequality by –2 will get rid of the fraction, but it will also reverse the direction of the inequality.

$x + 1 \leq -8$.

Subtracting one to each side of this inequality will make x the subject of the inequality:

$x \leq -9$.

The range of numerical values that satisfy the inequality is the set of numbers that are less than or equal to –9.

c) When manipulating the inequality,

$2 < 3x + 1 < 11$,

you need to be careful to perform the manipulations on the left side, the middle and the right side of the inequality. Like the previous cases, the objective is to isolate x.

Subtracting one:

$2 - 1 < 3x < 11 - 1$

$1 < 3x < 10$.

Dividing by 3:

$\frac{1}{3} < x < \frac{10}{3}$.

The range of numerical values that satisfy the inequality is the set of numbers that are greater than 1/3, but less than 10/3, including neither 1/3 nor 10/3.

More complicated inequalities

Manipulating inequalities may involve raising both sides of an inequality to a power, or manipulating fractions that have x in the denominator. Manipulating these more complicated inequalities is possible with the following rules.

Suppose that a and b are positive real numbers with $a < b$. Then:

$a^2 < b^2$

$\frac{1}{a} > \frac{1}{b}$.

One way to remember the second rule is that when you “flip” fractions, you reverse the direction of the inequality, i.e. if

$$\frac{a}{1} < \frac{b}{1}$$

then

$$\frac{1}{a} > \frac{1}{b}.$$

Example H.3

Use words to describe the range of numerical values that work in each of the following inequalities.

a) $0 < \dfrac{1}{x+3} \leq 4$.

b) $1 < \sqrt{3-x} < 2$.

c) $\dfrac{x-3}{\sqrt{x}} > 1$.

Solution:

a) There are two things to be careful of here. Firstly, in order to isolate *x*, it will be necessary to "flip" the fraction,

$$\frac{1}{x+3},$$

and you will need to remember that "flipping" a fraction reverses the direction of the inequality. Secondly, the left side of the inequality is zero. This cannot be "flipped," as it impossible to have zero in the denominator of a fraction.

To get around this, you can break this into two inequalities and consider them separately.

To ensure that $0 < \dfrac{1}{x+3}$, it is enough to ensure that $0 < x + 3$, and this is accomplished when $x > -3$.

To ensure that $\dfrac{1}{x+3} \leq 4$, "flip" both sides of this inequality and reverse the direction of the inequality:

$x + 3 \geq 1/4$

$x \geq 13/4$.

So, the range of numerical values that work in the inequality $0 < \frac{1}{x+3} \leq 4$ are all numbers that are greater than or equal to 13/4.

b) Since all of the numbers and quantities involved in the inequality are positive, squaring the numbers and quantities will not alter the directions of the inequalities. Squaring all of the parts of the inequality:

$1 < 3 - x < 4$

$-2 < -x < 1$

$2 > x > -1.$

(Note that multiplying through by –1 has reversed the directions of the inequalities.)

So, all numerical values between –1 and 2 will work in the inequality.

c) As a first manipulation designed to get rid of the fraction, you could multiply both sides of the inequality by $\sqrt{x}$:

$x - 3 > \sqrt{x}.$

This is valid, so long as the square root is positive. The next manipulation would be to square both sides of the inequality to eliminate the square root:

$(x - 3)^2 > x.$

Next, multiplying out the brackets, collecting like terms gives:

$x^2 - 6x + 9 > x$

$x^2 - 7x + 9 > 0$

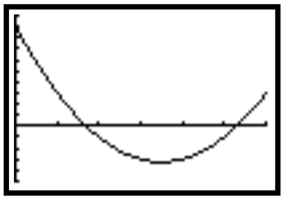
The x-values where $x^2 - 7x + 9 > 0$ can be determined by factoring and studying each factor (see later in this appendix), or by graphing

$y = x^2 - 7x + 9$

on a computer or calculator (see Figure H.0 below). The graph shows that $x^2 - 7x + 9 > 0$ when $x < 1.697$ and when $x > 5.30277$. The original inequality,

$$\frac{x-3}{\sqrt{x}} > 1$$

is satisfied when $x > 5.30277$.



$[0, 6]\times[-5, 10]$

Figure H.0: A graph of $y = x^2 - 7x + 9$.

Determining the sign of an algebraic expression

One of the most important applications of calculus is optimization. Optimization is the science of determining how a process can be most efficiently, profitably or cost-effectively run. Part of a typical optimization problem from calculus involves deciding where an algebraic expression is positive and where it is negative.

This process can be accomplished by factoring the algebraic expression as much as you possibly can, determining where each factor is positive and negative, and finally combining this information using the familiar rules for multiplying positive and negative numbers:

(*positive*)×(*positive*) = (*positive*)

(*positive*)×(*negative*) = (*negative*)

(*negative*)×(*negative*) = (*positive*)

Example H.4

Determine where the algebraic expression,

$x^3 + 3x^2 + 2x$

is positive, and where it is negative.

Solution:

We begin by factoring $x^3 + 3x^2 + 2x$ as much as is possible. Each term in this expression has a common factor of x, so you can factor that out:

$x^3 + 3x^2 + 2x = x(x^2 + 3x + 2)$.

The brackets contain a quadratic expression that can be factored in the usual fashion (see Appendix B and Appendix D) to:

$x^2 + 3x + 2 = (x + 1)(x + 2)$.

Thus,

$x^3 + 3x^2 + 2x = x(x + 1)(x + 2)$.

Next, the places where each of the three factors are positive and negative can be determined.

The first factor, x, is positive when $x > 0$, and negative when $x < 0$. The second factor, $x + 1$, is positive when $x > -1$ and negative when $x < -1$. Lastly, the third factor, $x + 2$, is positive when $x > -2$, and negative when $x < -2$.

Number lines can be a useful device for organizing the information about the three factors (see Figure H.1 below). The information about the expression $x^3 + 3x^2 + 2x$ can be deduced by reading down the number lines, and applying the rules for multiplying positive and negative numbers.

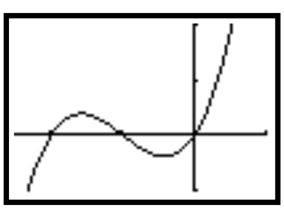
x	– – –	– – –	– – –	+ + +
$x + 1$	– – –	– – –	+ + +	+ + +
$x + 2$	– – –	+ + +	+ + +	+ + +
Product of the three	– – –	+ + +	– – –	+ + +
		–2	–1	0

Figure H.1: Number lines for determining the sign of $x^3 + 3x^2 + 2x$.

So, the conclusion is that the algebraic expression $x^3 + 3x^2 + 2x$ is:

- positive when $-2 < x < -1$ and when $x > 0$, and,
- negative when $x < -2$ and when $-1 < x < 0$.

You can check these conclusions by graphing the algebraic expression $y = x^3 + 3x^2 + 2x$ on a computer or calculator (see Figure H.2 below).



[–2.5, 1]×[–1, 2]

Figure H.2: The graph of $y = x^3 + 3x^2 + 2x$.

In today's world where you will have ready access to graphing calculators and computers, the ability to deduce where an algebraic expression has positive numerical values and where it has negative numerical values is probably not as important as it used to be. After all, you can just graph a formula and see for yourself where the graph is above the x-axis (this is where the formula is positive) and where the graph is below the x-axis (this is where the formula is negative). The kind of analysis presented here is still important for three reasons.

1. If you don't have a calculator available to you for some reason, then you can still determine the sign of an algebraic expression.

2. The skills you develop here can help you to build an "intuitive" sense of how algebraic expressions behave. This intuition can help to guide your efforts to analyze mathematical expressions and problems more effectively.

3. You can double-check the answers that you obtain with a calculator or computer - especially if the algebraic expression is one that a calculator is not able to display very well.

Interval notation

An easy way to write ranges of numerical values is to use to use interval notation. A closed interval $[a, b]$ indicates all real numbers x for which $a \le x \le b$. Closed intervals include their endpoints. An open interval (a, b) indicates all real numbers x for which $a < x < b$. Open intervals exclude their endpoints. Half-open or half-closed intervals are denoted by $(a, b]$ or $[a, b)$. These notations indicate all real numbers x for which $a < x \le b$ (that is, open at a and closed at b) and $a \le x < b$ (open at b and closed at a) respectively.

Caution: The notation (3, 4) can be interpreted as the point (3, 4) or as the open interval consisting of all numbers strictly between 3 and 4. Context determines the meaning.

Exercises for Appendix H

For Problems 1-10, use words to describe the range of numerical values that will satisfy the given inequality.

1. $x + 3 > 0$.

2. $2y - 1 < 2$.

3. $\dfrac{1}{3x+1} < 2$.

4. $\sqrt{u-1} \ge 3$.

5. $|x| < 9$.

6. $\dfrac{1}{x^2} > 4$.

7. $\dfrac{1-x}{x^2+7} \le 0$.

8. $w^2 + 2w + 1 < 0$.

9. $-3(p + 7) > 0$.

10. $2 + \dfrac{1}{x} > 0$.

For Problems 11-20, use words to describe the range of numerical values that will satisfy the given inequality.

11. $0 < \dfrac{x}{\sqrt{1+x^2}} < 1$.

12. $-1 < 4x + 3 < 1$.

13. $1 < \sqrt{1+x^2} < 2$.

14. $-2 < \dfrac{1-2u^2}{1+2u^2} < 3$.

15. $0 < \dfrac{1}{x+1} - 2 < 1$.

16. $0 < t^2 + t - 2$.

17. $0 < \sqrt{\dfrac{L}{32}} < 1$.

18. $-6 < |y+5| < 10$.

19. $\dfrac{1}{x^2} > \dfrac{1}{4x-4}$.

20. $\sqrt{x^2+1} < \sqrt{1-x}$.

For Problems 21-25, decide where each of the following algebraic expressions is positive and where each of the following algebraic expressions is negative.

21. $w^2 - 4$.

22. $u^3 - u$.

23. $e^t \cdot (t + 1)$.

24. $\dfrac{r+2}{r-1}$.

25. $\dfrac{y^2+3y+2}{y+3}$.

Answers to Exercises for Appendix H

1. All real numbers greater than –3.
2. All real numbers less than 3/2.
3. All real numbers greater than –1/2.
4. All real numbers greater than or equal to 10.
5. All real numbers between –9 and +9 (not including either –9 or +9).
6. All real numbers between –1/2 and +1/2, excluding –1/2, +1/2 and zero.
7. All real numbers greater than or equal to 1.
8. There are no real numbers that satisfy this inequality.
9. All real numbers less than –7.
10. All real numbers greater than zero, and all real numbers less than -1/2.
11. All real numbers greater than 0.
12. All real numbers greater than –1 and less than –1/2.
13. All real numbers greater than 0 and less than 1.
14. All real numbers.
15. All real numbers greater than 1/3 and less than 1/2.
16. All real numbers greater than 1 and all real numbers less than –2.
17. All real numbers greater than 0 and less than 32.
18. All real numbers greater than –15 and less than 5.
19. All real numbers less than 1.
20. All real numbers greater than –1 and less than zero.
21. Positive when $w < -2$ or when $w > 2$. Negative when $-2 < w < 2$.
22. Positive when $-1 < u < 0$, and when $u > 1$. Negative when $u < -1$ and when $0 < u < 1$.
23. Positive when $t > -1$. Negative when $t < -1$.
24. Positive when $r > 1$ and when $r < -2$. Negative when $-2 < r < 1$.
25. Positive when $-3 < y < -2$ and when $y > -1$. Negative when $y < -3$ and when $-2 < y < -1$.