Homework Assignment 18: Solutions

1. Using the short-cut rules for differentiation, $f(x) = x^3 - 5x + 1$ can be differentiated without all of the steps and manipulations that the limit definition requires. The derivative of f is given by the equation:

$$f'(x) = 3x^2 - 5.$$

To locate the places where the graph of the original function $y = f(x)$ has a hill-top (maximum) or valley-bottom (minimum) value, you can set the derivative equal to zero and solve for x. (The reason that this technique works is shown in Figure 1 below.)

At the top of a hill (maximum) the tangent line is horizontal.

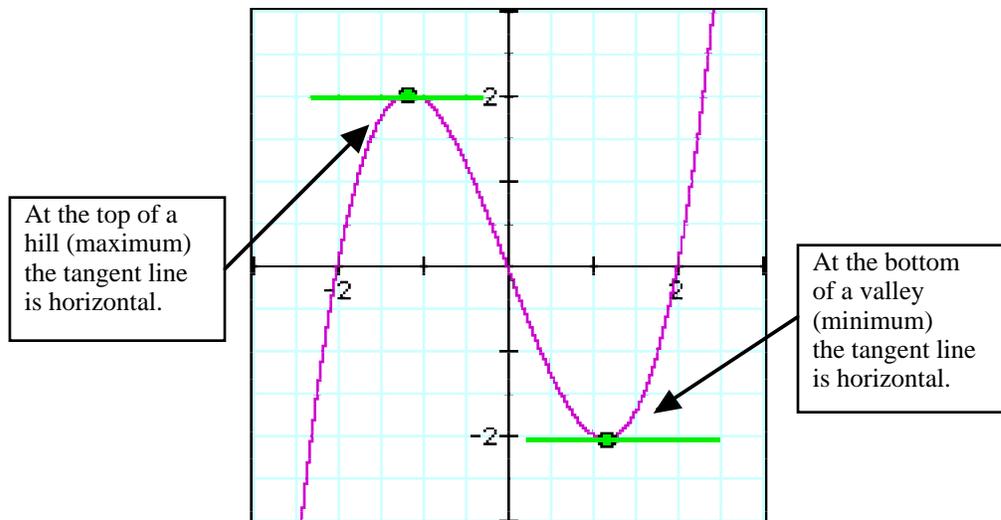At the bottom of a valley (minimum) the tangent line is horizontal.

Figure 1: Behavior of the derivative and tangent line at the top of a hill (maximum) or bottom of a valley (minimum).

Solving the equation $f'(x) = 0$ for x:

$$3x^2 - 5 = 0$$

$$3x^2 = 5$$

$$x^2 = \frac{5}{3}$$

$$x = \pm\sqrt{\frac{5}{3}}.$$

This gives the x-coordinates of the hill-tops (maximums) and valley-bottoms (minimums). To find the y-coordinates of these points, you can substitute each value of x into the equation for $f(x)$. This gives the x and y-coordinates of each point:

- $(-\sqrt{\frac{5}{3}}, 5.5033)$

- $(\sqrt{\frac{5}{3}}, -3.3033)$.

To determine whether a given point is a hill-top (maximum) or valley-bottom (minimum) you can locate the points on the graph of $y = f(x)$ (see Figure 2).
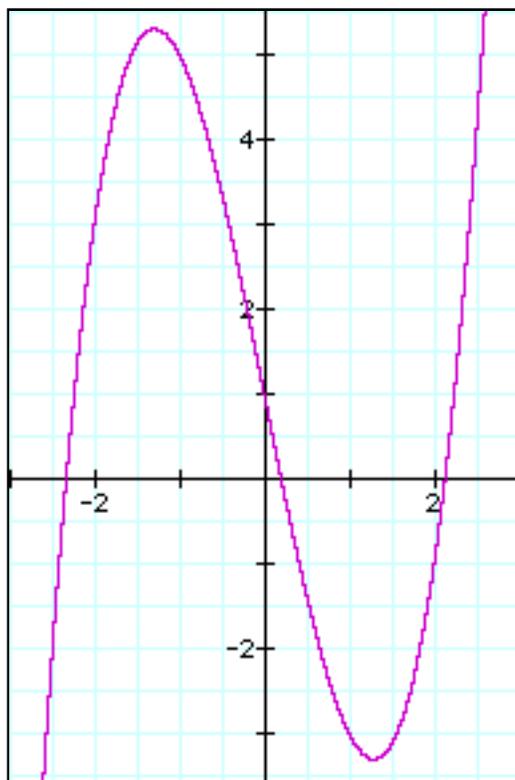

Figure 2: Graph of $y = f(x)$.

If a graph of $y = f(x)$ is not readily available, you can also check the sign of the derivative just to the left and just to the right of the point where $f'(x) = 0$. The pattern of "+" and "−" can reveal whether the point is a maximum or a minimum (see Figure 3).

Just to the left of the maximum, the slope of the tangent line is positive.

Just to the right of the maximum, the slope of the tangent line is negative.

Just to the left of the minimum, the slope of the tangent line is negative.

Just to the right of the minimum, the slope of the tangent line is positive.
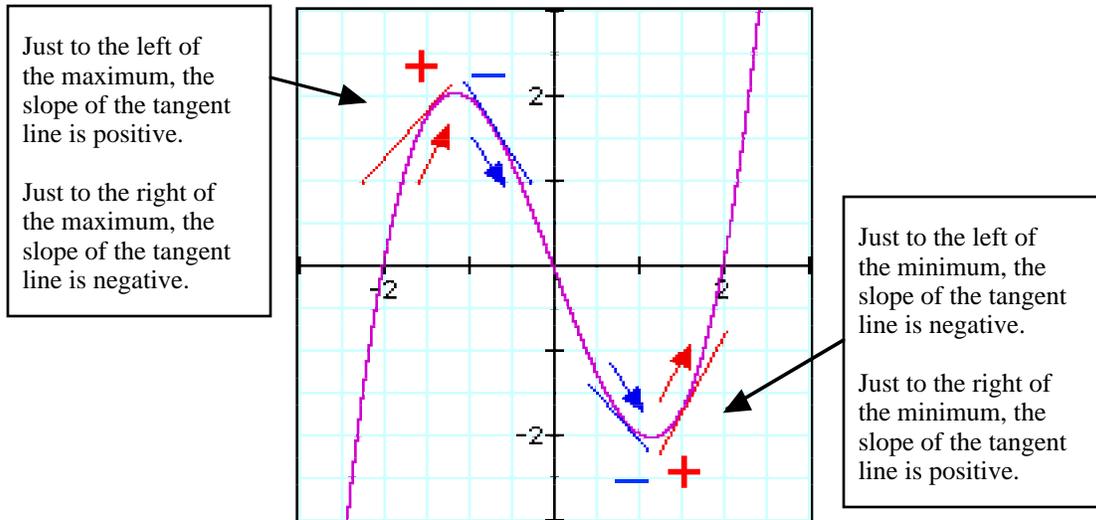
Figure 3: The relationship between the sign of the derivative just to the left and just to the right of the point where the derivative is equal to zero and the nature of the point (maximum or minimum).

Performing the calculations for the points obtained here gives the following:

x-coordinate	$f'(x)$ just to the left of the point	$f'(x)$ just to the right of the point	Type of Point
$x = -\sqrt{\dfrac{5}{3}}$	+0.07	−0.0077	Hill-top (Maximum)
$x = \sqrt{\dfrac{5}{3}}$	−0.0077	0.07	Valley-bottom (Minimum)

2. There are two immediate interpretations of what the question is asking here. However, only one interpretation will be suitable for use later in the homework assignment.

The first interpretation is that the question is asking you to find a function for the amount of energy that a squirrel expends while actually foraging. As we are told that a 150g squirrel expends 13.695 kJ per hour while foraging, this would suggest a function of the form:

$$C_{first}(T) = 13.695 \cdot T.$$

Where this interpretation is not so useful is that it does not take into account that a 150g squirrel expends 2.73 kJ of energy per hour even when resting. This suggests a second interpretation for the function - the amount of energy that the squirrel expends foraging plus the amount of energy that the squirrel expends while resting. If the squirrel forages for T hours then it will rest for $24 - T$ hours before foraging again. The total amount of energy expended by the squirrel is then:

$$C(T) = 13.695 \cdot T + 2.73 \cdot (24 - T) = 65.52 + 10.965 \cdot T.$$

This second interpretation is the one that will make sense when used later in the homework assignment.

The problem domain will be the set of possible values of T that make sense as potential squirrel foraging times. As squirrels typically only spend 7.5 hours above ground each day, and only forage when above ground, the appropriate problem domain would be:

$$0 \le T \le 7.5.$$

3. **Energy Gained by a Squirrel Eating Fungus:**

Each piece of fungus takes the squirrel a total of 13 minutes to locate and consume (10 to locate, 3 to consume). So in one hour, a squirrel can locate and consume:

$$\frac{60}{13} = 4.6154 \quad \text{fungi.}$$

Each piece of fungus yields 17.3 kJ, of which the squirrel is able to extract 52.2%. In energy terms, each hour the squirrel is able to obtain:

$$\frac{60}{13} * 17.3 * 0.522 = 41.6798 \quad \text{kJ.}$$

Therefore, if the squirrel forages for a total of T hours, then the squirrel will gain:

$$R_{fungus}(T) = 41.6798 * T \quad \text{kJ.}$$

Energy Gained by a Squirrel Eating Seeds:

Each cone of seeds (pine or fir) takes the squirrel a total of 33 minutes to locate and consume (18 to locate, 15 to consume). So in one hour, a squirrel can locate and consume:

$$\frac{60}{33} = 1.818 \quad \text{fungi.}$$

Each cone of seeds yields 26.05 kJ, of which the squirrel is able to extract 96%. In energy terms, each hour the squirrel is able to obtain:

$$\frac{60}{33} * 26.05 * 0.96 = 45.4691 \quad \text{kJ.}$$

Therefore, if the squirrel forages for a total of T hours, then the squirrel will gain:

$$R_{seeds}(T) = 45.4691 * T \quad \text{kJ.}$$

4.   The calculations and conclusions for each individual case are presented one-by-one below. The results of these calculations are summarized in the following table.

Case	Amount of time squirrel should spend foraging (hours)
1	2.13
2	7.5
3	1.899
4	7.5

Case 1: It is Summer and the squirrel feeds mainly on fungus.

The squirrel needs to consume enough kJ of energy to offset the amount of energy that it expends each day. That is, the time that the squirrel should spend foraging is the value of T that solves the equation:

$$R_{fungus}(T) = C(T).$$

Solving for T:

$$41.6798 * T = 65.52 + 10.965 * T$$

$$T = \frac{65.52}{30.7148} = 2.13 \text{ hours.}$$

Case 2: It is Fall and the squirrel feeds mainly on fungus.

Here the squirrel needs to maximize the net amount of energy that it gains. In financial terms, if the energy that the squirrel gains from foraging is likened to revenue and the energy that the squirrel expends is likened to costs, then the squirrel wants to maximize profits. Writing $P_{fungus}(T)$ for the net energy gained by the squirrel each day:

$$P_{fungus}(T) = R_{fungus}(T) - C(T) = 30.7148 * T - 65.52.$$

The squirrel wants to make this as large as possible. However, if you take the derivative of $P_{fungus}(T)$ then you get 30.7148 which cannot possibly be set equal to zero. Another approach is needed.

Graphing $y = P_{fungus}(T)$ for the problem domain of $0 \le T \le 7.5$ gives the graph shown in Figure 4 below.
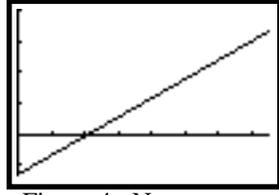
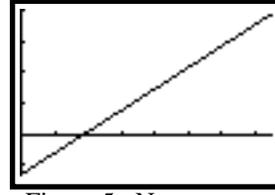Figure 4: Net energy
graph for a squirrel
consuming fungus.

Figure 5: Net energy
graph for a squirrel
consuming seeds.

Figure 4 shows that $P_{fungus}(T)$ is maximized when T is allowed to be as large as possible - in this case, $T = 7.5$. Hence, the squirrel should spend 7.5 hours foraging each day.

Case 3: It is Summer and the squirrel feeds mainly on seeds.

The squirrel needs to consume enough kJ of energy to offset the amount of energy that it expends each day. That is, the time that the squirrel should spend foraging is the value of T that solves the equation:

$$R_{seeds}(T) = C(T).$$

Solving for T:

$$45.4691 * T = 65.52 + 10.965 * T$$

$$T = \frac{65.52}{34.5041} = 1.899 \text{ hours.}$$

Case 4: It is Fall and the squirrel feeds mainly on seeds.

Here the squirrel needs to maximize the net amount of energy that it gains. In financial terms, if the energy that the squirrel gains from foraging is likened to revenue and the energy that the squirrel expends is likened to costs, then the squirrel wants to maximize profits. Writing $P_{seeds}(T)$ for the net energy gained by the squirrel each day:

$$P_{seeds}(T) = R_{seeds}(T) - C(T) = 34.5041 * T - 65.52.$$

The squirrel wants to make this as large as possible. However, if you take the derivative of $P_{seeds}(T)$ then you get 34.5041 which cannot possibly be set equal to zero. Another approach is needed.

Graphing $y = P_{seedss}(T)$ for the problem domain of $0 \le T \le 7.5$ gives the graph shown in Figure 5 (above). Figure 5 shows that $P_{seeds}(T)$ is maximized when T is allowed to be as large as possible - in this case, $T = 7.5$. Hence, the squirrel should spend 7.5 hours foraging each day.

5. The graphs for squirrels on a diet consisting mainly of fungus are shown below.
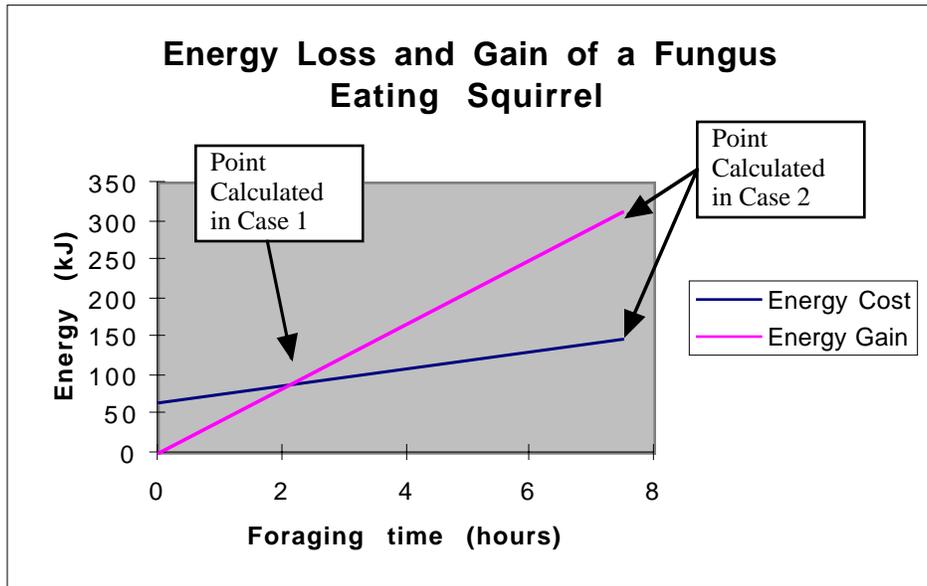

Figure 6: Fungus eating squirrel

The graphs for squirrels on a diet consisting mainly of seeds are shown below.
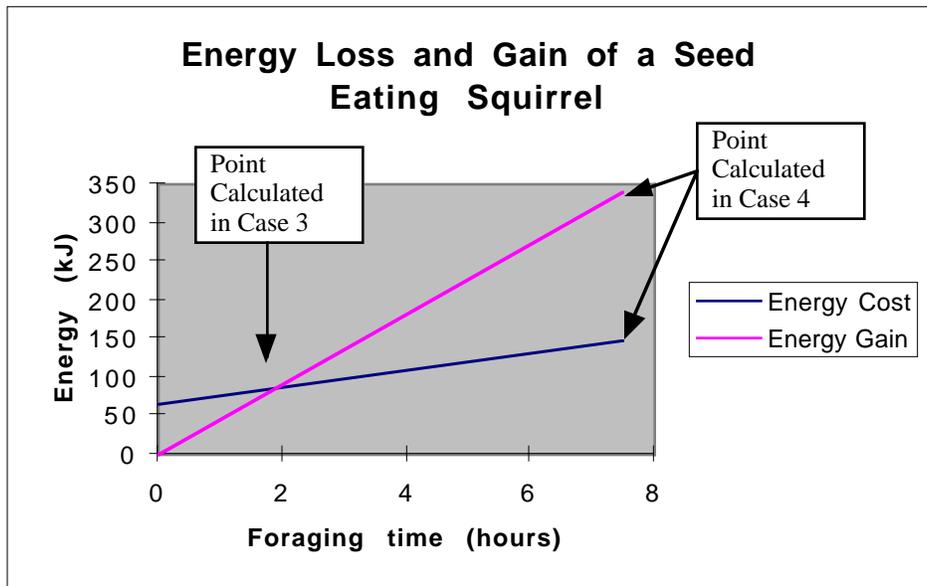

Figure 7: Seed eating squirrel

From these graphs it seems to be the case that:

"If x is restricted to an interval $[a, b]$, then the maximum and minimum values of a function $f(x)$ occur either:

- at a point where $f'(x) = 0$, or,
- at one of the endpoints ($x = a$ or $x = b$) of the interval $[a, b]$."