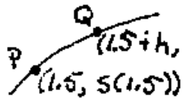
Solutions to First Exam from Oct. 26, 1998

1. Part I.

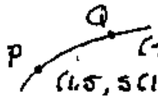
(a) 3 $s(1.5)$ gives the trolley's position at 1:30

(b) 1 We're looking for $s'(1.5)$ (2) is NOT equal to $s'(1.5)$ since h is heading towards 1.5



If Q moves towards P h must tend towards zero.

(1) gives $s'(1.5)$



As $x \to 1.5$ $Q \to P$, as desired.

(c) 8

(d) 9 Average velocity $= \frac{\Delta \text{position}}{\Delta \text{time}} = \frac{\Delta s}{\Delta t} = \frac{s(1.5)-s(0)}{1.5}$

(e) 5, 7 Change in velocity = velocity at 1.5 − velocity at 0 $= s'(1.5) - s'(0)$

(f) 6 Speed at 1:30 $= |s'(1.5)| = \left| \lim_{\Delta t \to 0} \frac{s(1.5+\Delta t) - s(1.5)}{1.5} \right|$

Part II.

(a) At what time is the trolley 2 miles south of the city center?

When is position equal to −2?

$s(t) = -2$ Solve for t.

(b) When is the trolley going south at 2 mph?

When is velocity equal to −2?

$s'(t) = -2$ Solve for t

Common Error Some people wrote $|s'(t)| = 2$. This corresponds to "when is the speed = 2". Speed = 2 when velocity is 2 OR −2, i.e. when the trolley is going 2 mph north or 2 mph south.

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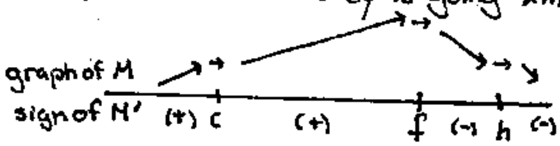
2. (a) f Where $M' \geq 0$, $M(t)$ is increasing



(b) e The population is increasing most rapidly where M' is greatest

(c) g The population is decreasing most rapidly where M' is most negative.

(d) c, f, h $M(t)$ has a horizontal tangent line wherever $M'(t) = 0$

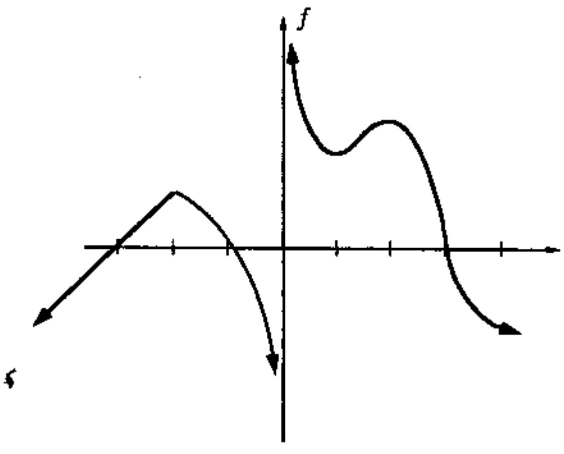
$M'(t) = 0$ at $t = c, f$, and h

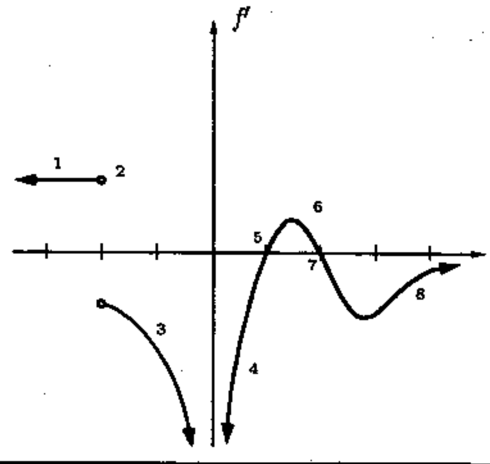
At b, e, and g $M'(t)$ has a horizontal tangent line – but $M(t)$ does NOT!

At c and h both $M(t)$ and $M'(t)$ have horizontal tangent lines.

(e) Yes, $M'(t)$ is a function, since for every value of t there is 1 distinct value of $M'(t)$ – in other words, at each time t there is exactly one slope for $M(t)$. $M'(t)$ passes the vertical line test.

Problem 4





Problem 4 continued

Annotations

1. For $x < -2$, the derivative is positive and constant (the function is linear with positive slope).

2. At -2, the derivative is undefined (the function is not locally linear).

3. For $-2 < x < 0$, the derivative is negative and decreasing, starting out at a negative value going to $-\infty$ at $x \to 0$ (the function is decreasing and concave down).

4. For $0 < x < 1$, the derivative is increasing but negative, approaching $-\infty$ as $x \to 0$ (the function is decreasing and concave up).

5. At $x = 1$, the derivative is zero (the tangent line is horizontal).

6. For $1 < x < 2$, the derivative is positive; it increases between 1 and (roughly) 1.5, and decreases for the remainder of this interval.

7. At $x = 2$, the derivative is zero.

8. For $x > 2$, the derivative is negative; it decreases between 2 and (roughly) 2.8, then increases, approaching 0 as $x \to \infty$.

#5. a. Domain is all reals except $x=1$ (which gives division by zero).

b. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$ Note that $f(x+h) \neq \underbrace{\frac{x}{x-1} + h}_{}$; this was a common error. ← this is actually $f(x)+h$

$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$ Get a common denominator.

$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x^2+hx-h-x)-(x^2+hx-x)}{(x+h-1)(x-1)}$ Do NOT multiply out the denominator. You'll just have to refactor it later!!

$= \lim_{h\to 0} \frac{-h}{h(x+h-1)(x-1)}$ Watch out for your negative signs. Many people lost the $-$ on $-h$.

$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \boxed{\frac{-1}{(x-1)^2}}$

c. Since the denom. of $f'(x)$ is squared, it's never negative. Thus, $f'(x)$ is always negative since it has a -1 on the top.
If $f'(x)$ is negative, $f(x)$ must be decreasing.

d. $f'(3) = \frac{-1}{(3-1)^2} = -\frac{1}{4}$ Don't try to do this from scratch with the limit definition! You already have a formula for $f'(x)$.

To check, use a secant line through a nearby point.

$f'(3) \approx \frac{f(3.01)-f(3)}{.01} = \frac{\frac{3.01}{2.01} - \frac{3}{2}}{.01} = -.248756\ldots$

(5) continued

e. Equation means $y = mx + b$.

$m = f'(3) = -1/4$ from (d)

The pt. is $(3, f(3)) = (3, 3/2)$. Note: can't use $(3, f'(3))$.

Solve for b:

$y = -\frac{1}{4}x + b$

$3/2 = -\frac{1}{4} \cdot 3 + b \Rightarrow b = 9/4$

$\Rightarrow y = -\frac{1}{4}x + \frac{9}{4}$

6. 1. D 2. A
3. F 4. C
5. E 6. B

Common Error: mixing up 1 and 6: Look at the extremities – for $|x|$ large, graph 1 has a slope close to zero while graph 6 is decreasing and steep.

Problem 7

(b). $P(x)$ is linear, and so is relatively easy to write down; it's just

$$P(x) = 20 + .1x .$$

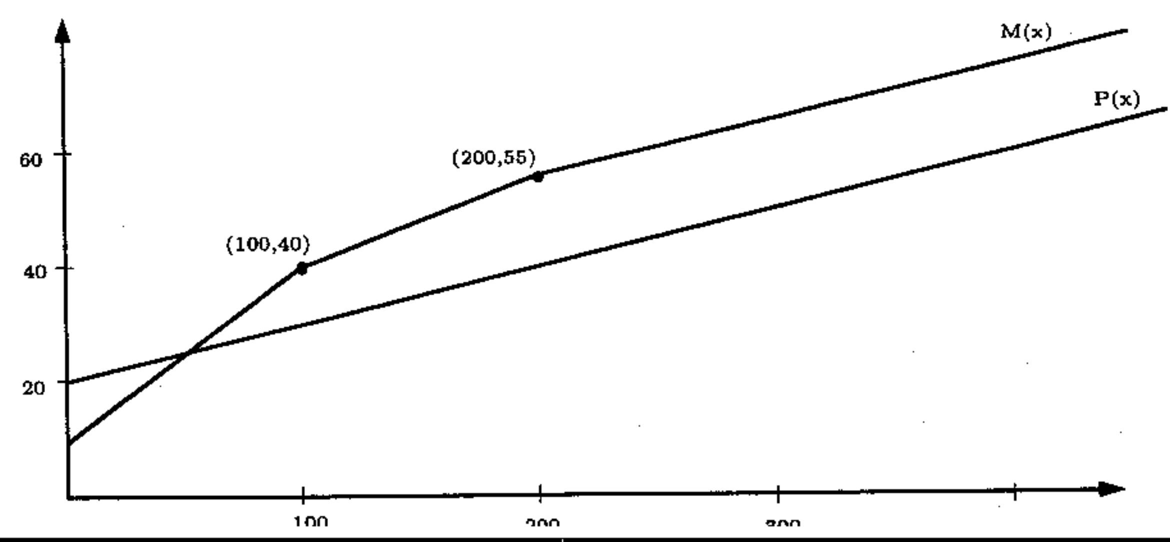
$M(x)$ is only piecewise linear, and so has to be expressed in cases; it's

$$M(x) = \begin{cases} 10 + .3x, & \text{if } x \le 100; \\ 40 + .15(x - 100), & \text{if } 100 \le x \le 200; \text{ and} \\ 55 + .1(x - 200), & \text{if } x \ge 200 \end{cases}$$

or, equivalently, if we multiply out,

$$M(x) = \begin{cases} 10 + .3x, & \text{if } x \le 100; \\ 25 + .15x, & \text{if } 100 \le x \le 200; \text{ and} \\ 35 + .1x, & \text{if } x \ge 200 \end{cases}$$

(a). The graphs of P and M look like:



(7) continued

(c). The two graphs cross somewhere between $x = 0$ and $x = 100$. To find out exactly where, we find for what value of x in this range we have $P(x) = M(x)$: that is,

$$20 + .1x = 10 + .3x.$$

Solving, we find

$$x = 50.$$

Thus, it's cheaper to get the pens from Pens-R-Us whenever we want to buy 50 or more pens.

(d). Note that the two graphs are parallel for $x \geq 200$ (that is, the two companies charge the same amount per pen beyond the 200$^{\text{th}}$ pen). The maximum difference in price is simply the difference $M(x) - P(x)$ for any $x \geq 200$, which is \$15.

8. (a) The domain is $[100, 600]$.

$$\text{altitude of ride} \xrightarrow{C} \text{cost}$$

The balloonist will give rides that take you to a height of between 100 ft and 600 ft.

Note: it doesn't make logical sense to say that the balloon itself never goes under 100 ft!

(b) $C(200) = 30$

For \$30 the balloonist will give a ride up to an altitude of 200 feet.

(c) $C'(200) = .15$

At an altitude of 200 feet, the cost of the ride is increasing at a rate of 15¢/foot.

Alternatively,

Once you've taken a ride up to 200 feet, going up an additional foot would cost approximately an additional 15¢.

Common Errors:

We are given an instantaneous rate of change, so we cannot say that after reaching 200 feet it will cost 15¢ for every additional foot. (E.g. $C'(201)$ may be .16. Nor do we know the cost of each foot up to 200 feet from $C'(200) = .15$.

We are ONLY being given information about the instantaneous rate of change of cost with respect to altitude reached.

Note that velocity does not play a role – we have no information about velocity.

(d) $C'(A) > 0$ for all $A > 100$

For all rides to altitudes over 100 ft, the price of the ride increases with the maximum altitude reached.

In other words, cost increases with altitude for altitudes over 100 ft.

Note: $C'(A) > 0$ is NOT the same as $C'(A)$ increasing.

$C'(A) > 0$ means $C(A)$ is increasing (cost increases with altitude)

whereas $\underbrace{C'(A) \text{ increasing}}_{\text{we don't know if this is the case!}}$ means $C(A)$ is concave up – the cost/foot is increasing.