Math Xa Fall '99
First Exam: Annotated Solutions

1 a) • (i)
• (ii)

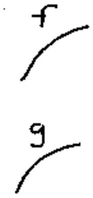
b) • (i)
• (ii)

c) • (iv)
• (ii)
• (i)

Comments

a) f is positive, increasing and concave down
- positive → No info about f'
- increasing → $f' > 0$
- concave down → f' decreasing

b) g is negative, increasing and concave down
- negative → No inf about g'
- increasing → $g' > 0$
- concave down → g' decreasing



where a function is increasing, its derivative is positive
where a function is decreasing, its derivative is negative
where a function is concave down, its derivative is decreasing
where a function is concave up its derivative is increasing

c) h' is negative, increasing, and concave down
- negative → h decreasing
- increasing → h concave up
- concave down → Not of much interest to us - no basic inf about h.

2. a)
$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{5(3+h)}{2(3+h)+4} - \frac{5\cdot 3}{2\cdot 3+4}}{h}$$
$$= \lim_{h\to 0} \left[\frac{15+5h}{6+2h+4} - \frac{15}{10}\right]\cdot\frac{1}{h}$$

Common Error: $\frac{A}{h} \neq A\cdot\frac{h}{1}$; $\frac{A}{h} = A\cdot\frac{1}{h}$

$$= \lim_{h\to 0} \left[\frac{15+5h}{10+2h} - \frac{3}{2}\right]\cdot\frac{1}{h}$$

Get a common denominator

$$= \lim_{h\to 0} \left[\frac{15+5h}{2(5+h)} - \frac{3(5+h)}{2(5+h)}\right]\cdot\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{15+5h-15-3h}{2(5+h)}\cdot\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{2h}{2(5+h)}\cdot\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{1}{5+h} = \frac{1}{5} = .2$$

b) Average rate of change of f over $[2.99, 3]$
$$= \frac{f(3)-f(2.99)}{3-2.99} = \frac{\frac{5\cdot 3}{2\cdot 3+4} - \frac{5(2.99)}{2(2.99)+4}}{.01} = \frac{\frac{15}{10} - \frac{14.95}{9.98}}{.01} \approx .2004008\ldots$$

Comment: Your answer to (b) should be close (very close) to your answer to (a). If they aren't close you need to look for an error.

Recall: def'n: Average rate of change of f over $[a,b] = \frac{f(b)-f(a)}{b-a}$. Some people made this unnecessarily difficult!

3. a) at $t=3$

graph of T ↗ → ↘
sign of T' + 3 −

b) When is T' greatest? Answer: at $t=-1$.

c) When is T' most negative? Answer: at $t=4$.

d) On $[7,8]$ T' is negative so T is decreasing. T' is a negative constant, so T is decreasing at a constant rate.

Common Errors

(a) T is greatest at $t=3$. The greatest value of T is $T(3)$.

(b,c) The answers should be a value of t, NOT an interval.
For instance, at $t=4$, the graph of T is horizontal. It is NOT increasing rapidly.
T' increasing doesn't imply T increasing!!

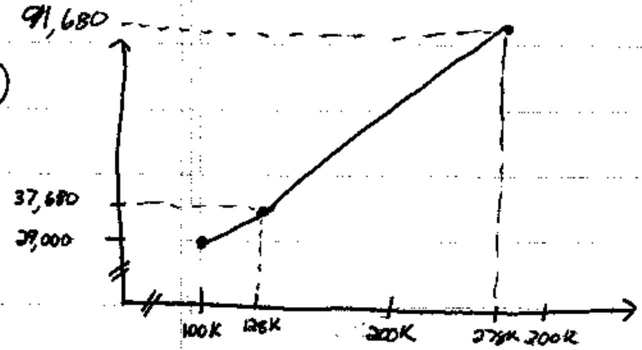
4

a) $T(x) = \begin{cases} 29{,}000 + .31(x - 100{,}000) & 100{,}000 \le x < 128{,}000 \\ 29{,}000 + .31(28{,}000) + .36(x - 128{,}000) & 128{,}000 \le x \le 278{,}000 \end{cases}$

NOTE: there are a number of different combinations of $<$ + $\le$ that correctly model the function

COMMON ERRORS: Some people didn't model the tax rates as <u>marginal</u> rates, i.e. they wrote equations where the rate applied to the <u>entire</u> income, not just income over the breaking points.

b.)



c) $T(200{,}000) = 37{,}680 + .36(200{,}000 - 128{,}000)$
$= 63{,}600$

This is more than twice the tax paid by someone earning \$109,000

COMMON ERRORS: starting the graph at 0 instead of 100K
making the 2nd piece less steep than the first piece

d. $T'(120{,}000) = .31$. This means that the tax rate for an income of \$120K is 31% + the <u>next dollar earned</u> will mean an additional \$.31 in taxes owed

e. $T'(150{,}000) = .36$. At an income of \$150K, the tax rate is 36%. I.e., the <u>next dollar earned</u> will mean an additional \$.36 in taxes owed.

COMMON ERRORS: Some people described T' as the "instantaneous rate of change" but did not <u>interpret</u> what this meant in practical terms.

5\. a) VIII
b) VI
c) IV
d) II.

a) Common Error: for large x the value of the function appproaches a positive number, but the slope approaches zero.

7\. Reupholstering rates:

Chairs: $\frac{8 \text{ hrs}}{C \text{ chairs}} = 8/C$ hrs./chair

couches $\frac{D \text{ hours}}{1 \text{ couch}} = D$ hours/couch

painting: $S \frac{\text{sq.ft.}}{\text{hr.}} \rightarrow 1/S \frac{\text{hrs.}}{\text{sq. Foot}}$

time to paint:

A sq.ft $\longrightarrow A \text{ sq.ft} \frac{1}{S} \frac{\text{hrs}}{\text{sq.ft.}} = \frac{A}{S}$ hours

X chairs $\longrightarrow X \text{chairs} \cdot \frac{8}{C} \frac{\text{hrs.}}{\text{chair}} = \frac{8X}{C}$ hours

Y couches $\longrightarrow Y \text{couches} \cdot D \frac{\text{hrs.}}{\text{couch}} = YD$ hours

total time: $\frac{A}{S} + \frac{8X}{C} + YD$ hours.

COMMON ERROR: dividing instead of multiplying or vice versa. Checking units is a good way to avoid this pitfall

b) Hours needed for Q chairs:

$Q \text{ chairs} \cdot \frac{8}{X} \frac{\text{hrs.}}{\text{chair}} = \frac{8Q}{X}$ hours.

Hours already spent: T hours

Hours remaining: $\frac{8Q}{X} - T$ hours