Solutions to Midterm 2 Math Xa Fall 2000 Nov. 30 00

(1) Use product, quotient rule appropriately (note derivative of e^{2x} is $2e^{2x}$). Note – no need to simplify your answers!

(a) could use quotient rule, or, better yet, simplify a bit first:

$$f(x) = \frac{1+2x^3-8x^5}{4x^4} = \frac{1}{4x^4} + \frac{2x^3}{4x^4} - \frac{8x^5}{4x^4} = \frac{1}{4}x^{-4} + \frac{1}{2}x^{-1} - 2x$$

so $f'(x) = -\frac{4}{4}x^{-5} - \frac{1}{2}x^{-2} - 2 = -x^{-5} - \frac{1}{2}x^{-2} - 2$

(b) $g'(x) = 5x^4 + 4x\,(2e^{2x}) + e^{2x}\,(4) = 5x^4 + 8xe^{2x} + 4e^{2x}$

(c) $h'(x) = \dfrac{(x^3-x^2)e^x - (e^x-2)(3x^2-2x)}{(x^3-x^2)^2} + 437\pi$

(2) (a) critical points of $f(x)$ are where $f'(x)=0$ (since we'll see that $f'(x)$ is always defined, and there are no domain endpoints)

$f'(x) = x^2 \cdot e^x + e^x \cdot 2x + 0 = (x^2+2x)e^x$

Note $e^x > 0$ for all x, so $f'(x)=0$ when $x^2+2x=0$, or when $x(x+2)=0$, i.e. when $x=0$ or -2

values of $f(x)$ at these critical points are $f(0) = 0^2e^0+2 = 2$ and $f(-2) = (-2)^2e^{-2}+2 = \frac{4}{e^2}+2$

(b) to find extrema, just need to check the two critical points $x=0$ and -2. To help out, first use a number line to investigate sign of $f'(x)$:

$f'(x)$ +++0 – –0 ++

$f' = x(x+2)e^x$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2$

for $x>0$ $f'>0$

$-2<x<0$ $f'<0$

↗ ↘ ↗

this shows $f(-2)$ is a local max and $f(0)$ is a local min.

Now investigate more closely to determine whether these are absolute min or maxs. Since $f(0)=2$, and $f(x)=x^2e^x+2$ since we know $x^2e^x \geq 0$ for all x, then $x^2e^x+2 \geq 2$ for all x, so $f(0)=2$ is an absolute minimum (nothing gives a lower value than 2 for $f(x)$). On the other hand, it's easy to check that $f(x)$ can be greater than $f(-2)=\frac{4}{e^2}+2$ → for instance

(2) continued

$f(100) = 100^2 e^{100} + 2 > \frac{4}{e^2} + 2$ (by a large amount)

or just note $\lim_{x \to \infty} x^2 e^x + 2 = +\infty$, so there can't be an absolute max, so $f(-2) = \frac{4}{e^2} + 2$ is a local max.

(c) Inflection points are where $f(x)$ changes concavity, i.e. where $f''(x)$ changes sign. Since $f'(x) = (x^2 + 2x) e^x$

$$f''(x) = (x^2+2x)e^x + e^x(2x+2) = (x^2+4x+2)e^x$$

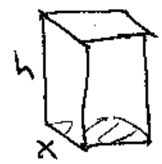
Again, since $e^x > 0$, then $f''(x)$ has roots and changes sign exactly at the two roots of x^2+4x+2, i.e. at $x = \frac{-4 \pm \sqrt{16-8}}{2} = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$

i.e. it has two inflection points, one at $x = -2+\sqrt{2}$
one at $x = -2-\sqrt{2}$

(3) (a)
(i) 1
(ii) 0
(iii) undefined
(iv) ∞
(v) $-\infty$
(vi) -1
(vii) 1
(viii) undefined

(b) $f'(x)$ is undefined wherever $f(x)$ has "sharp corners," is discontinuous, or is undefined, so at $x = -3, -2, 1, 3$ and 4

(4) (a) Note the two dimensions are height h, and base length x. Volume = base × height



$= x^2 h$
$= 100$ (this is given)

Cost depends on surface area.
Area of sides $= 4xh$, area of base $= x^2$, so factoring in \$2 per square foot cost for side glass, and \$10 per square foot for base yields

$$\text{Cost} = 2 \cdot 4xh + 10 \cdot x^2 = 8xh + 10x^2.$$

Now turn into function of one variable by solving for h in the equation for volume: $100 = x^2 h$
so $h = 100/x^2$

Thus $\text{cost} = 8x\left(\frac{100}{x^2}\right) + 10x^2$
$= \frac{800}{x} + 10x^2$ or $800x^{-1} + 10x^2$

To minimize cost, find derivative, set equal to zero
$C(x) = 800x^{-1} + 10x^2$, so $C'(x) = -800 \cdot x^{-2} + 20x$

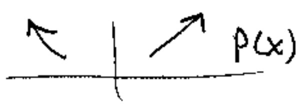
(4) continued

So now solve $0 = -800x^{-2} + 20x$, so $800x^{-2} = 20x$
or $800 = 20x^3$
or $40 = x^3$

Thus to minimize cost choose $x = \sqrt[3]{40}$,
and solve for height $h = \frac{100}{x^2} = \frac{100}{(\sqrt[3]{40})^2}$

(b) Is this really a minimum?
check with 2nd derivative test:
$C''(x) = 1600x^{-3} + 20 = \frac{1600}{x^3} + 20$
so $C''(\sqrt[3]{40}) = \frac{1600}{(\sqrt[3]{40})^3} + 20 > 0 \Rightarrow$ concave up
$\Rightarrow$ minimum ✓

(5) (a) $P(x)$ does not have any roots so FALSE because ...
(b) $P(x)$ is always positive - TRUE ($P(x)$ has only even degree terms, all positive coefficients $\rightarrow$ has to always be positive)
(c) TRUE - $P'(x)$ is degree 5. Odd degree implies at least one root,
(d) TRUE $P'(x)$ is degree 5 $\rightarrow$ at most 5 roots
(e) TRUE $P(x)$ has even degree so "arms" go in same directions - positive leading coefficient means arms up
(f) FALSE - since $\lim_{x\to\infty} P(x) = +\infty$, it can't have an absolute max.
(g) TRUE $\rightarrow$ same logic since $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = \infty$,



at some point it reached its lowest value $\Rightarrow$ global minimum

(h) TRUE - all even powers $\Rightarrow$ even function
check $P(-x) = a(-x)^6 + b(-x)^4 + c(-x)^2 + d$
$= ax^6 + bx^4 + cx^2 + d = P(x)$
$\Rightarrow$ even function.

(6) (a) from symmetry about $x=3$ line, clear that second point is $(6, c)$
(b) double root at $x=3$ implies $A(x) = k(x-3)^2$ for some scalar constant k. To find k, we

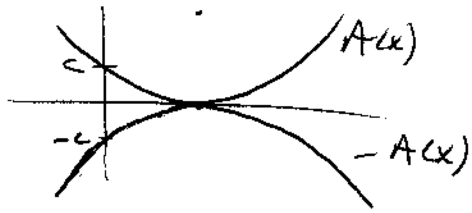
(6) (b) continued

see that $A(0)=c$, so $A(0)=k(0-3)^2=9k$

so $9k=c$, or $k=\frac{c}{9}$, Thus $A(x)=\frac{c}{9}(x-3)^2$

Note - many students thought they could "solve" for c. Sure, if you assume $A(x)=(x-3)^2$ then $A(0)=9$, but there's no reason to assume $A(x)$ isn't $2(x-3)^2$ or $3(x-3)^2$ for example, and these would give different results for c

(c) To Find $B(x)$, just flip $A(x)$:



and then shift up by $2c$

so $B(x) = -A(x)+2c$

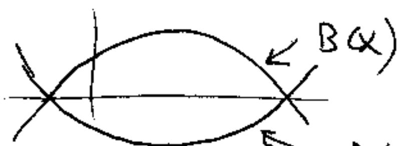
$= -\frac{c}{9}(x-3)^2+2c$

(d) For $C(x)$ to have the same derivative as $A(x)$ means that $C(x)$ is just an up or downward shift (translation) of $A(x)$. to get

a downward shift of c,

try $C(x)=A(x)-c = \frac{c}{9}(x-3)^2-c$

(e) Easy!



$D(x)=-B(x)=\frac{c}{9}(x-3)^2-2c$

Note: to find $-B(x)$, you need to be careful with your minus signs! $-B(x)\neq \frac{c}{9}(x-3)^2 \underline{\underline{+}} 2c$

(7) (a) the two datapoints for $C(t)$ are

when $t=0$, total complaints $=2$, so $C(0)=2$

when $t=50$ miles, total complaints $=3$, so $C(50)=3$

If $C(t)=C_0\cdot b^t$, then $C(0)=C\cdot b^0=C_0$
$=2$

so $C_0=2$

$C(50)=3$ implies $3=2\cdot b^{50}$, so $\frac{3}{2}=b^{50}$,

$b=\left(\frac{3}{2}\right)^{1/50}$, so $C(t)=2\cdot\left(\left(\frac{3}{2}\right)^{1/50}\right)^t$

or $2\cdot\left(\frac{3}{2}\right)^{t/50}$

(7) continued

(b) check $C(750) = 2\cdot\left(\frac{3}{2}\right)^{750/50} \approx 875.78$

which is very close to 875, so this seems compatible.

(c) Since for exponential functions $C'(x) = k\cdot C(x)$ for some constant k, then $\frac{C'(x)}{C(x)} = k$, the same for any value of x for the exponential function.

We know $C'(500) = 0.41$, and $C(500) = 2\cdot\left(\frac{3}{2}\right)^{\frac{500}{50}}$

also $C(750) = 2\cdot\left(\frac{3}{2}\right)^{750/50} = 2\cdot\left(\frac{3}{2}\right)^{15}$

so $\frac{0.41}{2\left(\frac{3}{2}\right)^{10}} = \frac{C'(750)}{2\cdot\left(\frac{3}{2}\right)^{15}}$, so $C'(750) = 0.41\left(\frac{2\cdot\left(\frac{3}{2}\right)^{15}}{2\left(\frac{3}{2}\right)^{10}}\right)$

$= 0.41\cdot\left(\frac{3}{2}\right)^{5}$

≈ 3.11 complaints/mile.

[Note, as we'll soon be able to check, 0.41 was not correct – this was a typo on the test, the result of changing some of the other numbers in the question. $C'(500)$ actually equals $\ln\left[\left(\frac{3}{2}\right)^{1/50}\right]\cdot 2\cdot\left(\frac{3}{2}\right)^{\frac{500}{50}}$

≈ 0.94

Of course we accepted all the answers everyone gave for full or partial credit as appropriate based on the erroneous value of $C'(500) = 0.41$. The actual complaints per mile when $t = 750$ (i.e. $C'(750)$) is ≈ 7.1

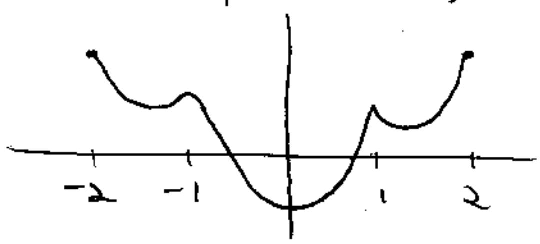
(8) (a) So $g(x) = kx(x-a)(x+a) = kx(x^2-a^2) = kx^3 - ka^2x$ to have roots at $x = -a, a$ and 0. so $g'(x) = 3kx^2 - ka^2$

Then $g''(x) = (6k)x$ (Note k and a are just numbers, constants, so be careful when taking derivatives $g'(x) \neq 3kx^2 - 2akx - ka^2$

so $g''(0) = 0$, and g'' changes from neg to pos at $x = 0$,

so $g(x)$ has an inflection point there, at $x = 0$, $g(0) = 0$

(b) Many variants:



domain $= [-2, 2]$
local max at $x = -1$
global min at $x = 0$
undefined deriv. at $x = 1$