Fall 1999

1. a) $\frac{d}{dx}\left(\frac{\pi}{(3^{5x})^2}\right) = \frac{d}{dx}\left(\pi \cdot 3^{-10x}\right) = \frac{d}{dx}\left(\pi \cdot (3^{-10})^x\right)$

$= \pi \ln(3^{-10}) \cdot (3^{-10})^x = \boxed{-10\pi \ln(3) \cdot 3^{-10x}}$

b) $\frac{d}{dx}\left(\frac{5e^{2x+1}}{7x^3}\right) = \frac{d}{dx}\left(\frac{5e}{7} \cdot e^{2x} \cdot x^{-3}\right) = \frac{5e}{7}\left[e^{2x} \cdot \frac{d}{dx}(x^{-3}) + \frac{d}{dx}((e^2)^x) \cdot x^{-3}\right]$

$= \frac{5e}{7}\left[e^{2x} \cdot (-3)x^{-4} + \ln e^2 \cdot e^{2x} \cdot x^{-3}\right] = \frac{5e}{7}\left[\frac{-3e^{2x}}{x^4} + \frac{2e^{2x}}{x^3}\right]$

$= \frac{-15e^{2x+1}}{7x^4} + \frac{10e^{2x+1}}{7x^3} = \boxed{\frac{e^{2x+1}}{7x^4}(10x-15)}$

c) $\frac{d}{dx}\left[\frac{1}{4}\ln\left[\frac{x \cdot 2^x}{3x+1}\right]\right] = \frac{1}{4}\frac{d}{dx}\left[\ln\left[\frac{x \cdot 2^x}{3x+1}\right]\right] = \frac{1}{4} \cdot \frac{1}{\left[\frac{x \cdot 2^x}{3x+1}\right]}\left[\frac{d}{dx}\left(\frac{x \cdot 2^x}{3x+\right.\right.$

$= \frac{3x+1}{2^2 \cdot 2^x \cdot x}\left[\frac{d}{dx}(x) \cdot \frac{2^x}{3x+1} + x \cdot \frac{d}{dx}\left[\frac{2^x}{3x+1}\right]\right] = \frac{3x+1}{2^{x+2} \cdot x}\left[\frac{2^x}{3x+1} + \right.$

$\left. x \cdot \left(\frac{\ln 2 \cdot 2^x(3x+1) - 2^x \cdot 3}{(3x+1)^2}\right)\right]$

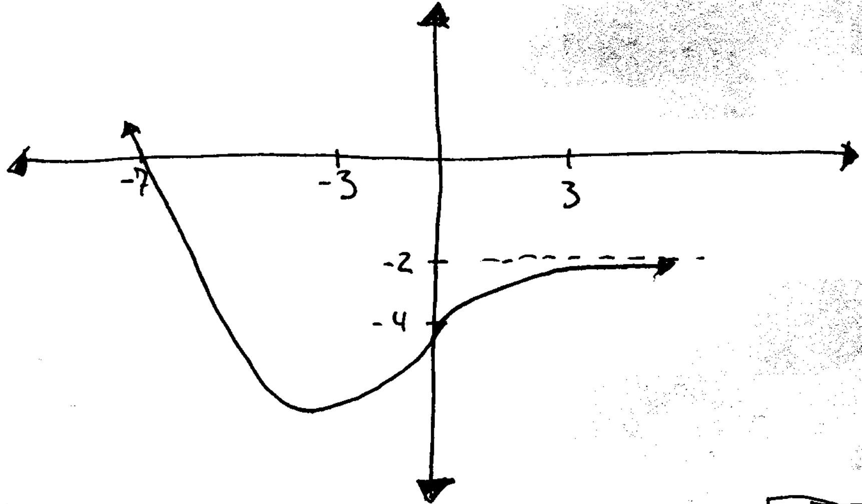
$= \frac{3x+1}{2^{x+2} \cdot x}\left[\frac{2^x}{3x+1} + \frac{\ln 2 \cdot 2^x \cdot x}{3x+1} - \frac{3x \cdot 2^x}{(3x+1)^2}\right] = \boxed{\frac{1}{4x} + \frac{\ln 2}{4} - \frac{3}{4(3x+1)}}$

d) $\frac{d}{dt}\left[\pi t \ln(t^3+3)\right] = \pi\left[\frac{d}{dt}(t) \cdot \ln(t^3+3) + t \cdot \frac{d}{dt}(\ln(t^3+3))\right]$

$= \pi \ln(t^3+3) + t \cdot \frac{1}{t^3+3} \cdot \frac{d}{dt}(t^3+3) = \pi \ln(t^3+3) + \frac{t \cdot 3t^2}{t^3+3}$

$= \boxed{\pi \ln(t^3+3) + \frac{3t^3}{t^3+3}}$

2.

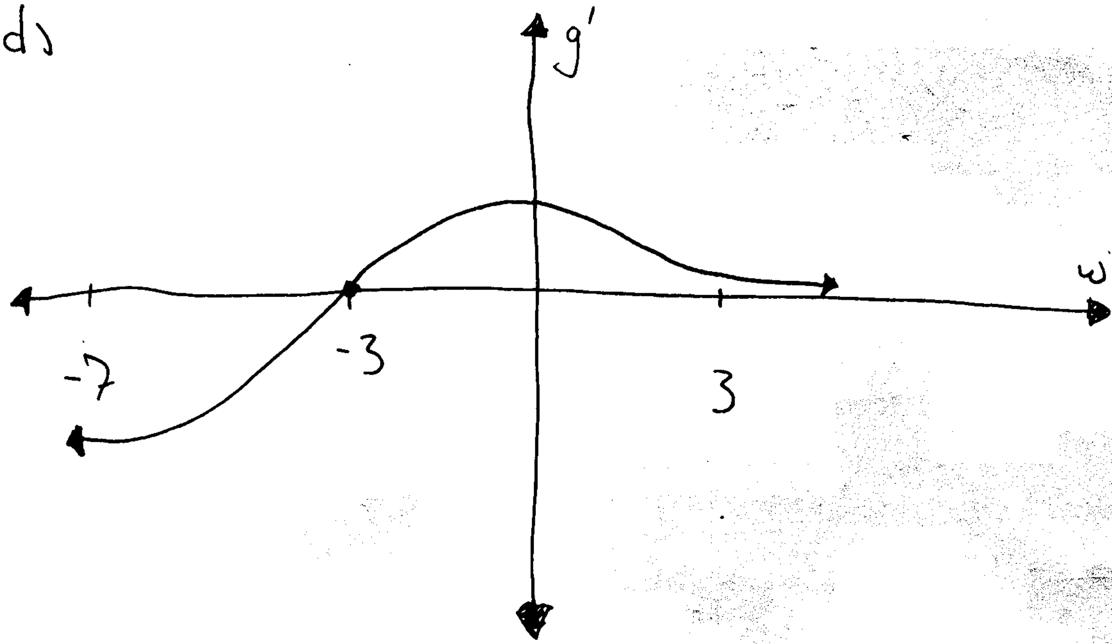


a) $g'(w)$ is positive from -3 to $+\infty$ $\Rightarrow$ $\boxed{(-3, \infty)}$

b) $g''(w)$ is positive from $-\infty$ to 0 $\Rightarrow$ $\boxed{(-\infty, 0)}$

c) $\lim_{w \to \infty} g'(w) = 0$ because the graph goes horizontal as $w \to \infty$.

d)



3. a) $f'(3) = \lim_{h\to 0}\left[\frac{f(3+h)-f(3)}{h}\right] = \lim_{h\to 0}\left[\frac{\frac{6+2h}{4+h} - \frac{6}{4}}{h}\right]$

$= \lim_{h\to 0}\left[\frac{24+8h-24-6h}{4h\cdot(4+h)}\right] = \lim_{h\to 0}\left[\frac{2h}{4h(4+h)}\right]$

$= \lim_{h\to 0}\left[\frac{1}{2(4+h)}\right] = \boxed{\frac{1}{8}}$

3b) $\frac{f(3.5)-f(3)}{.5} = \frac{\frac{7}{4.5}-\frac{6}{4}}{.5} = \frac{1}{9} \approx \frac{1}{8}$

Looks like the right ballpark. Confidence is good. The idea was to approximate the slope around a point by choosing 2 points close together - the closer the better.

4. a) population P 1st year $\rightarrow$ $P \cdot r$ 2nd year $\rightarrow$ $P \cdot r^2$ 3rd year $\rightarrow$ Pr^3 4th year. So after 3 years pass $Pr^3 = P \cdot 118\%$ where r is the amount grown each year (as a %age) $\Rightarrow r^3 = 1.18$

So $r = 1.056$ $\Rightarrow$ the population grew by less than 6%

$\Rightarrow \boxed{iii}$

b) $H_0 \cdot r^t$. $r^t = 118\%$ in 3 years $\Rightarrow 1.18^{\frac{t}{3}} = r^t$,

So $\boxed{H(t) = H_0 \, 1.18^{t/3}}$

c) $H(t) = H_0 \, 1.18^{t/3} = H_0 (1.25) \Rightarrow \frac{t}{3} = \log_{1.18} 1.25 \Rightarrow \boxed{t = \frac{3 \log 1.25}{\log 1.18}}$

$\approx \boxed{4.64 \text{ years}}$

d) $B_0 = \frac{H_0}{10}$ $\quad B(t) = B_0 \cdot 2^{\frac{t}{6}} = \frac{H_0}{10} \cdot 2^{t/6} = H_0 \cdot 1.18^{t/3}$

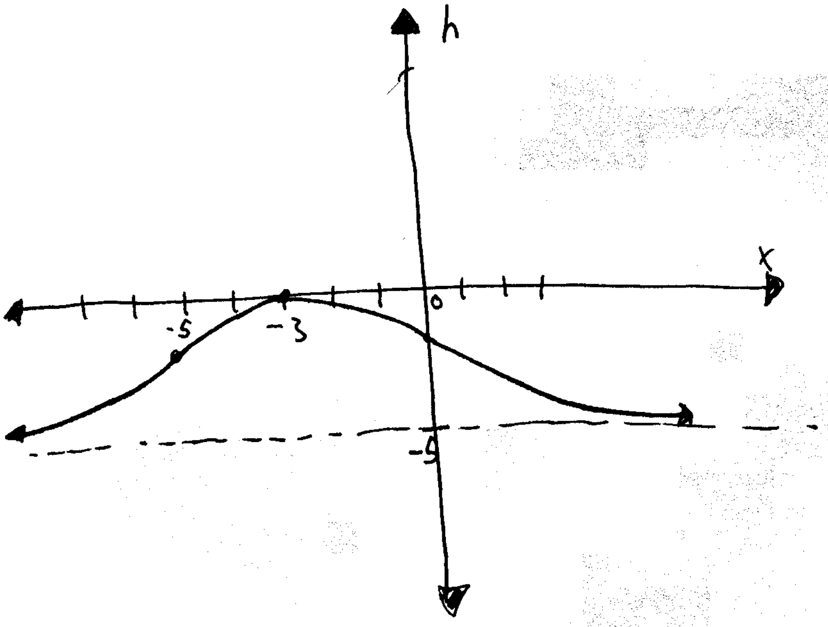
$\Rightarrow \frac{1}{10} 2^{t/6} = 1.18^{t/3} \Rightarrow \log\left(\frac{1}{10} \cdot 2^{t/6}\right) = \log\left(1.18^{t/3}\right) \Rightarrow$

$\log \frac{1}{10} + \log 2^{t/6} = \log\left(1.18^{t/3}\right) \Rightarrow \frac{t}{6} \log 2 - \log 10 = \frac{t}{3} \log 1.18$

$\Rightarrow t\left(\frac{1}{6} \log 2 - \frac{1}{3} \log 1.18\right) = \log 10 \Rightarrow \boxed{t = \frac{\log 10}{\frac{1}{6}\log 2 - \frac{1}{3}\log 1.18}}$

$\boxed{t \approx 38.15 \text{ years}}$

5. has 1 root ⇒ crosses x axis once. To cross only once, given the derivative, it must do it at the maximum point!



graph is steepest at -5 & 0.

6. a) i. positive → P gives the amount of paper, and always for a positive radius, so we have a positive quantity.

b) i. positive → the amount of paper on the roll increases as the radius increases.

c) $P'(4) = Q$ means that when the radius is 4 inches, the amount of paper is increasing by Q, or in physical terms, the circumference of the roll is Q inches.

d) $P^{-1}(w) = 2$ means that if there are w inches^2 of paper on the roll, then the radius is 2 inches.

7. a) $\boxed{f(x) = \frac{-3}{4}(x+2)^2}$ → 1 root, even power, and $f(0) = -3$ determines the constant out front.

b) $\boxed{f(x) = 3 \cdot 2^{\frac{-x}{3}}}$ → exponential, goes to 0 as $x \to \infty$, and $f(0) = 3$ determines the constant out front. $f(-3) = 6$ determines the base: $3 \cdot b^{-3} = 6 \Rightarrow b^{-3} = 2 \Rightarrow b = 2^{-1/3}$

8. a) $\lim_{x\to 0^+} \frac{|x|}{x} = \boxed{+1}$

b) $\lim_{x\to 0^-} \frac{|x|}{x} = \boxed{-1}$

c) $\lim_{x\to 0} \frac{|x|}{x}$ is undefined.

d) $\lim_{x\to 0} \frac{\ln(3+x) - \ln 3}{x} = \frac{d}{dx}[\ln x]$ @ $x=3 \Rightarrow \boxed{\frac{1}{3}}$

9. a) $h(x) = \ln(2x^5) = 5\ln(x) + \ln 2$ can only have values when $x > 0$. Since $\ln(x)$ is one-to-one for $x>0$, h has an inverse function.

b) if $h(x) = \ln(2x^5)$, then $x = \ln(2 \cdot h^{-1}(x)^5)$

$\Rightarrow e^x = 2 \cdot h^{-1}(x)^5 \Rightarrow \frac{e^x}{2} = h^{-1}(x)^5 \Rightarrow \boxed{\frac{e^{\frac{x}{5}}}{\sqrt[5]{2}} = h^{-1}(x)}$

10. a) Revenue R = Passengers P * fare f.

$P = 800 + \frac{20(99-f)}{2} = 800 + 990 - 10f = 1790 - 10f$

$R = P \cdot f = (1790 - 10f) \cdot f = 1790f - 10f^2$

$\frac{dR}{df} = 0 \Rightarrow 1790 - 20f = 0 \Rightarrow \boxed{f = \frac{179}{2}}$

but we have a max for P of $1000 \Rightarrow f$ minimum $= 79$

R @ $79 = 79000$, R @ $\frac{179}{2} = 159{,}310$

That's all our critical points, so $\boxed{f = \$\frac{179}{2}}$

b) $\boxed{R(\frac{179}{2}) = \$159{,}310}$

11. $f(x) = x \ln x$

a) $\ln x$ for $x \leq 0$ is undefined, so the domain of f is $\boxed{(0, \infty)}$

b) $f(x) = 0 \Rightarrow x = 0$ or $\ln x = 0$. $\ln x = 0$ if $x = 1$, so

$f(x) = 0$ at $\boxed{x = 0, 1}$

c) Critical points are where $f'(x) = 0$, undefined, or at $0, \infty$ (the endpoints).

$f'(x) = 0 \Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow \boxed{x = \frac{1}{e}}$

$f''(x) = \frac{1}{x}$, which is positive over our domain.

So $\frac{1}{e}$ is minimum (absolute), 0 is a local maximum, ∞ is an absolute maximum.

d) yes, there's an abs. minimum at $\boxed{x = \frac{1}{e}}$

e) there's a maximum value of ∞ as $x \to \infty$.