## Practice Problems: Test #2

Important Information:

1.   The first test will be held on Thursday November 29 from 7:00-9:00pm in Science Center Lecture Hall D..

2.   The test will include approximately eight problems (each with multiple parts).

3.   You will have 2 hours to complete the test.

4.   You may use your calculator and one page (8" by 11.5") of notes on the test.

5.   The specific topics that will be tested are:

- Quadratic functions
- Functions defined in pieces
- The idea of a limiting value
- Left and right hand limits
- Calculating limits of functions
- Horizontal and Vertical asymptotes
- Compound interest and the growth of investments
- Average and instantaneous rates of change
- The derivative function
- Calculating derivatives using the definition
- Graphical and verbal interpretations of the derivative
- Slope fields and solutions of differential equations
- Calculating derivatives with short-cut rules
- The product and quotient rules
- Polynomial, rational functions and their graphs
- Finding formulas for derivatives using the definition and regression
- Solving exponential equations using logarithms
- Derivatives of exponential and logarithmic functions
- The number 'e,' and derivative of $y = e^x$
- Locating the maximum, minimum values of a function
- Modeling situations with functions and optimization

6.   I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

7.   Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

8.   Good sources of help:

- Section leaders' office hours (posted on Xa web site).
- Math Question Center
- Labs on Tuesday 11/27 and Thursday 11/29.
- Course-wide review on Wednesday 11/28 from 8:30-10:00pm in Science Center 102B.

1.   In this problem, the function f is the function defined by the graph shown below.

As x decreases, the function oscillates back and forth.

$y = f(x)$
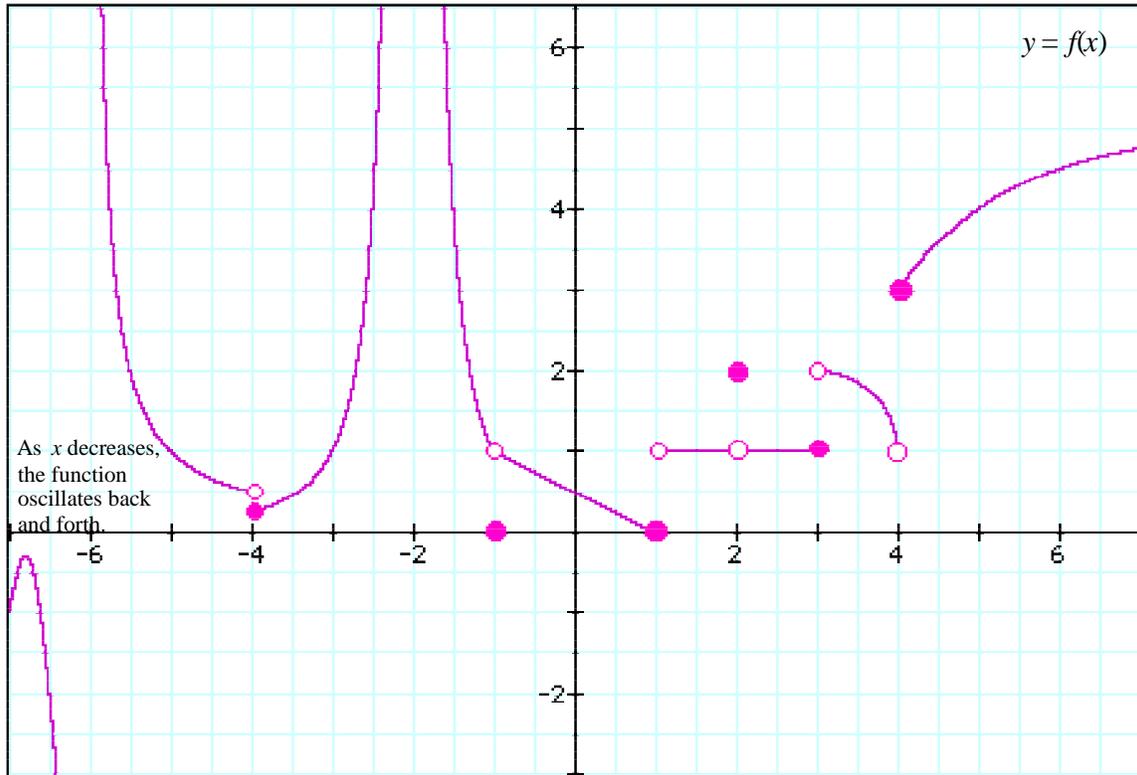
Evaluate the following limits. If the limit does not exist, explain why.

(a) $\underset{x \to +\infty}{Lim} f(x)$

(b) $\underset{x \to -\infty}{Lim} f(x)$

(c) $\underset{x \to -4}{Lim} f(x)$

(d) $\underset{x \to -6}{Lim} f(x)$

(e) $\underset{x \to -1}{Lim} f(x)$

(f) $\underset{x \to 1}{Lim} f(x)$

(g) $\underset{x \to 2}{Lim} f(x)$

(h) $\underset{x \to 2}{Lim} f(2)$

(i) $\underset{x \to -2}{Lim} f(x)$

2. A car starts moving down a straight road at $t = 0$. After traveling for t seconds, the car has covered $s(t)$ feet. The function s is defined by the equation given below.

$$s(t) = \begin{cases} 2.64 \cdot t^2 & ,0 \le t \le 20 \\ 105.6t + 1056 & ,20 < t \end{cases}$$

(a) What is the average speed over the interval [0, 10] ?

(b) What will the speedometer of the car read at the instant of time when $t = 10$?

(c) What will the speedometer of the car read at the instant of time when $t = 30$?

(d) What is the average speed of the car over the interval $[10, 10 + k]$ where k is a number between 0 and 10?

(e) What is the limit of the average speed of the car over the interval $[10, 10 + k]$ as $k \to 0^+$?

3. In this problem, the function f will always refer to the function defined by the graph shown below.
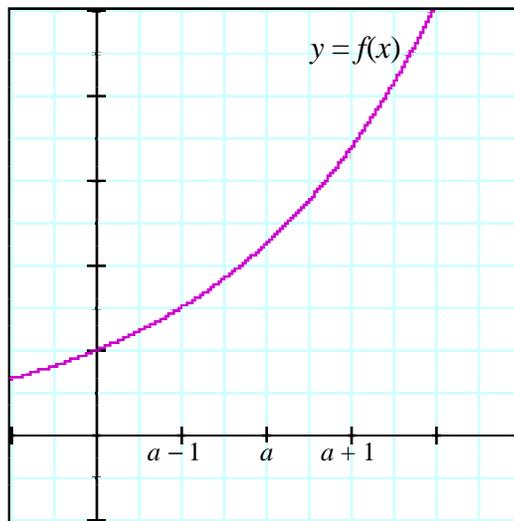
(a) Sketch the line that passes through the point $(a, f(a))$ and has slope equal to the derivative of f at $x = a$.

(b) Write down an equation (i.e. something along the lines of $y = m \cdot x + b$) for the line that you sketched in Part (a).

Note: Your equation may contain symbolic quantities such as a, $f(a)$, etc. You do not need to find numerical values for these symbolic quantities.

(c) Evaluate the linear function that you found in Part (b) at the points $x = a - 1$ and $x = a + 1$.

(d) Suppose that in this situation, x represents the amount of time that a bacterial colony has been growing, and $f(x)$ represents the number of bacteria in the colony after 'x' hours. Explain the practical significance of the symbolic statement:

$$f'(7) = 14$$

in terms of time and bacterial so that a person who was not familiar with calculus would have some idea of what you were talking about.

4.   A tomato is thrown vertically into the air at time $t = 0$. Its height, $d(t)$ (in feet), above the ground at time t (in seconds) is given by the equation:

$$d(t) = -16 \cdot t^2 + 48 \cdot t.$$

(a)   Sketch a graph showing $y = d(t)$.

(b)   Find the times when the tomato's height above the ground is equal to zero. What is happening to the tomato the first time $d(t) = 0$? What is happening the second time?

(c)   When does the tomato reach its maximum height above the ground?

(d)   What is the maximum height of the tomato above the ground?

(e)   The equation for $d(t)$ is a quadratic function that has been written in standard form. Use the technique of completing the square to write an equation for $d(t)$ that is in vertex form.

(f)   How could you have used your answers to Parts (b), (c) and (d) to determine the vertex form of the equation for $d(t)$?

5.   In this problem, assume that the functions f, g, h and j are all rational functions with equations of the form:

$$y = \frac{(x - A) \cdot (x - B)}{(x - C) \cdot (x - D)}$$

where A, B, C and D are all numbers. The graphs of the four functions are shown in the diagrams below.
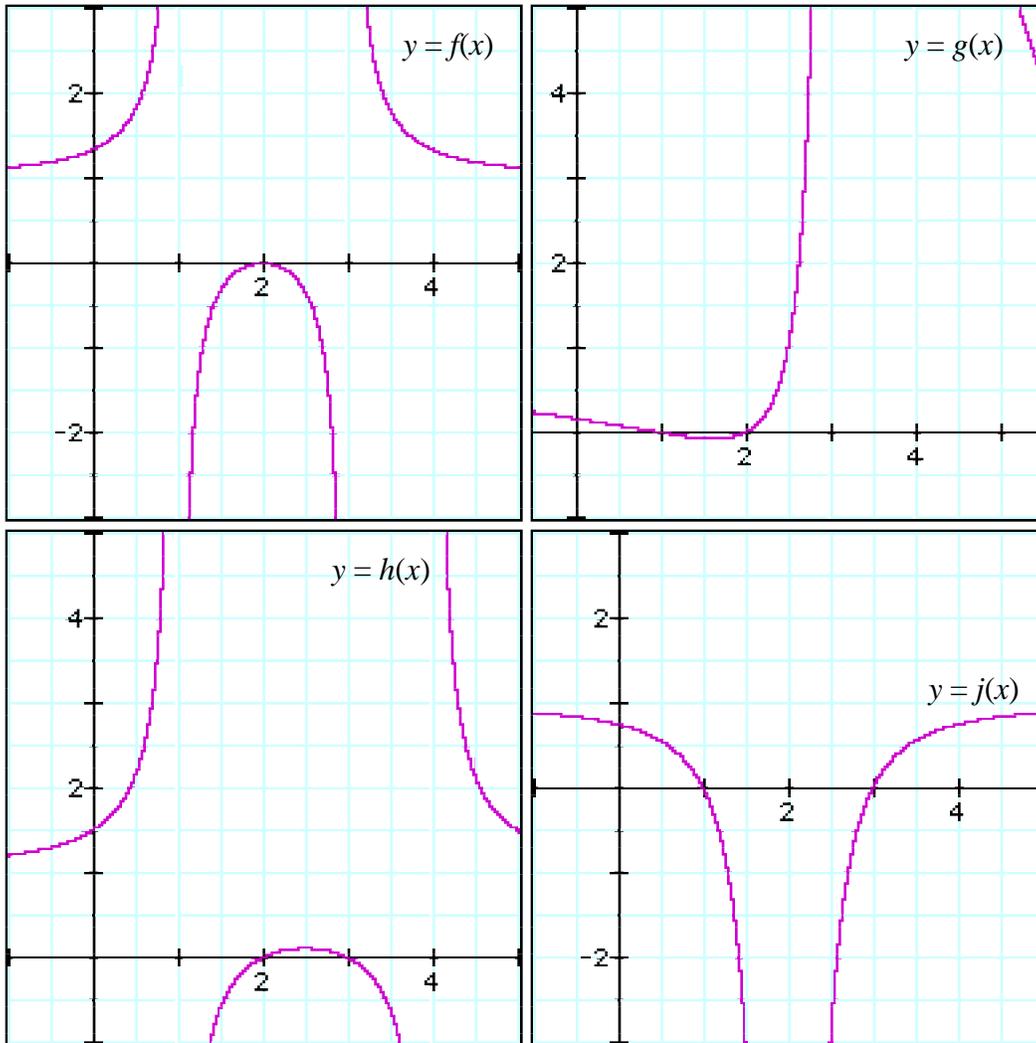
Each of the statements below shows how the numbers A, B, C and D are related to each other.   Decide which statement corresponds to which graph.

(a) $A < C$, $C = D$, $D < B$.

(b) $A < B < C < D$.

(c) $C < B < A < D$.

(d) $C < A$, $A = B$, $B < D$.

(e) $A < C < B < D$.

6.   The function f is defined by the graph shown below.   In this problem, assume that f is a polynomial with degree equal to 3.
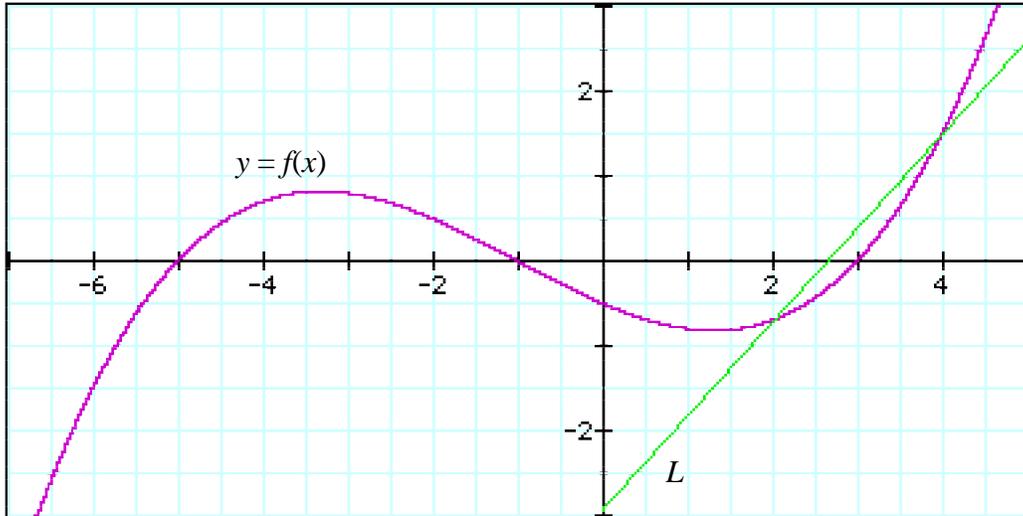
(a) Find an equation for the function f.

(b) Calculate the coordinates of the points where the function f and the line L intersect.

(c) Find an equation for the line L.

7. In this problem the function f will always refer to the function defined by the equation given below.

$$f(x) = \begin{cases} 2^x & ,0 \le x < 1 \\ x & ,1 \le x \le 2 \\ x^2 - 4 & ,2 < x \le 3 \end{cases}$$

(a) Plot a graph showing $y = f(x)$ for $0 \le x \le 3$.

(b) Find the numerical value of each of the following limits:

(I) $\displaystyle \lim_{x \to 1^+} f(x).$

(II) $\displaystyle \lim_{x \to 1^-} f(x).$

(III) $\displaystyle \lim_{x \to 2^+} f(x).$

**(IV)** $\displaystyle \lim_{x \to 2^-} f(x).$

(c) Often it is the case that the left hand limit or the right hand limit (sometimes both) is equal to the value of the function at a point. Does this always have to be the case? If you think so, explain why. If you don't think so, give an example to demonstrate your point of view.

8. In this problem, f is a function. All that you know about f is that:

- $f'(-1) = -2$
- $f'(0) = 0$
- $f'(1) = 2$

Which of the following graphs could possibly be the graph of $y = f(x)$?

(a)   **(b)**   **(c)**   **(d)**
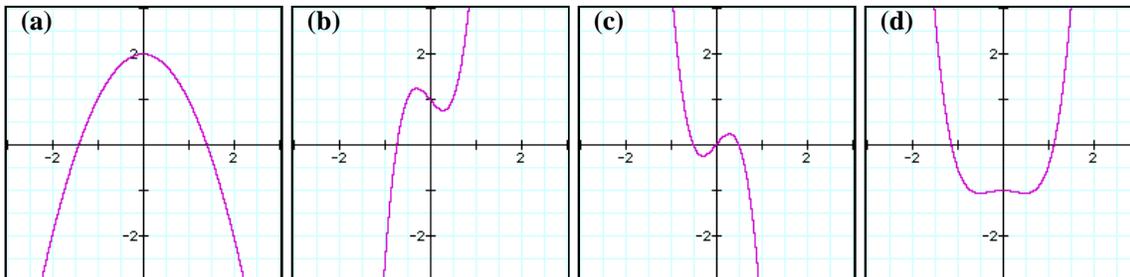
9. Organic matter contains a radioactive isotope of carbon, known as carbon-14. The half life of carbon-14 is 5730 years. A 100g sample of fresh organic matter will normally contain 0.0001 μg of carbon-14. In this problem, you will examine some of the ways in which the smallest mistakes can drastically alter results when exponential functions and logarithms are involved.

(a) Find an equation to describe the amount of carbon-14 (in micrograms) that remains in a 100g sample of organic matter after T years.

(b) Suppose that a 100g sample of organic matter contains 0.0000327 μg of carbon-14. How old is the organic matter?

(c) Now, suppose that you misread the number from Part (b) and used the figure of 0.0000372 μg of carbon-14 instead. How far off would the age of the organic matter be?

(d) Lastly, suppose that your little buddy Barry calculated the age of a sample of organic matter and determined that it was 2000 years old. Looking over Barry's work, you notice that he incorrectly remembered the half life of carbon-14 to be 5370 years and used this figure in his calculations. What is the correct age of the organic matter?

10. Hungary is one of the few countries in the world where the population is decreasing. In 1990 the population of Hungary was approximately 10.8 million. Since 1990, the population has been decreasing by approximately 0.2% every year.

(a) Find an equation that gives the population of Hungary as a function of time.

(b) Calculate the derivative of the function that you found in Part (a). What are the units of your derivative?

(c) At what rate was the population of Hungary changing at the beginning of the year 2000? Give a practical interpretation of this number that could be understood by a person who was not familiar with calculus.

(d) Find an equation for the inverse of the function from Part (a).

(e) Calculate the derivative of the inverse that you found in Part (d).

(f) Evaluate the derivative of the inverse when the population of Hungary is equal to the population of New Zealand (3.4 million).

(g) Give a practical interpretation of the numerical value that you obtained in Part (f).

11. In this problem the function f will always be the function defined by the equation given below.

$$f(x) = \frac{1}{1 + e^{-x}}.$$

(a) Determine the intervals on which f is an increasing function and the intervals on which f is a decreasing function.

(b) Is the inverse of f a function in its own right? If so, find a formula for the inverse of f.

12. "Yankee Home Furnishings" is a small, family-owned furniture company that specializes in hand-crafted furniture in a traditional New England style. Most of their business is with individual customers who want particular items of custom furniture made to their individual specifications. Occasionally, the company manufactures larger orders for corporate customers. In a major deal last year, the company agreed to deliver at least 300 and a maximum of 400 chairs for a major hotel chain. (The hotel chain was not sure of the exact number of chairs that they needed, and the deal was written so that they would get at least 300 chairs and more if they needed them.) The base price was $90 per chair up to 300 chairs. If the hotel chain wanted more than 300 chairs, then the price would be reduced by 25 cents per chair (on the whole order) for every additional chair over 300 that the hotel chain ordered. What is the largest and the smallest revenue that Yankee Home Furnishings could make from this deal?

13.(a) To help cover tuition and expenses, Marc works in a coffee shop. Through long and diligent practice, he has perfected the art of pouring coffee, so that the coffee runs into the cups at a constant rate. Match the graphs of the depth of coffee as a function of time with the four different shaped coffee cup. Be careful to explain why you think each choice is appropriate.
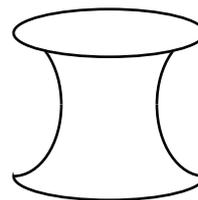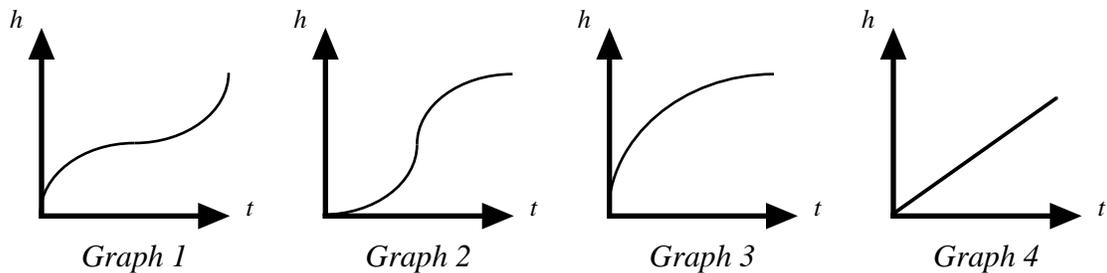

Cup 'A'   Cup 'B'   Cup 'C'   Cup 'D'

h

h

h

h

Graph 1   *Graph 2*   *Graph 3*   *Graph 4*
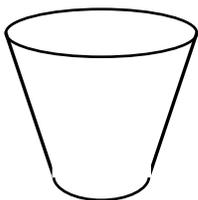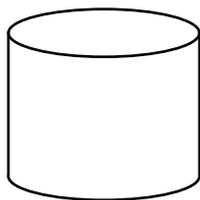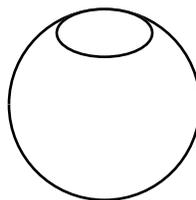
(b)   Let $d(t)$ be the depth of coffee (centimeters) in a coffee cup as a function of how much time Marc has been pouring for (in seconds). Write mathematical symbols that mean the same as:

> *It takes approximately 0.7 seconds for the depth of coffee in the cup to go from 10cm to 11cm.*

14.   In this problem, the functions f and g are defined by the equations given below.

$$f(x) = x^4 - x - 1 \qquad\qquad\qquad g(x) = \tfrac{1}{2}x^2 + 2$$

In this problem you will also be using a function h. All that you may assume about h is that:
- $h(2) = 1$
- $h'(2) = -1$.

(a)   Define a new function k by the equation: $k(x) = f(x) \cdot g(x)$. Find an equation for the derivative $k'(x)$.
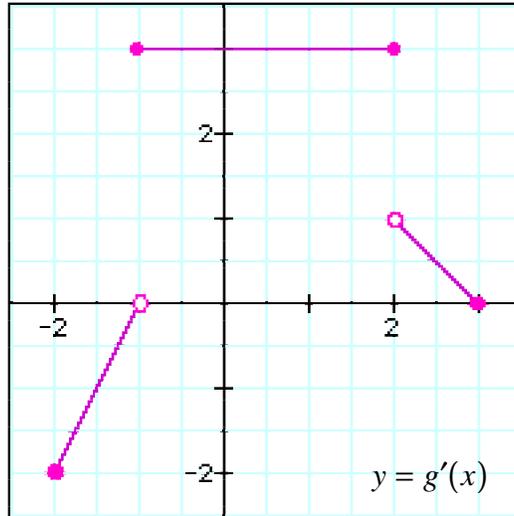
(b)   Define a new function m by the equation: $m(x) = \dfrac{g(x)}{h(x)}$. Calculate the numerical value of the derivative $m'(2)$.

(c)   Calculate an approximate value for $m(3)$.

15.   In this problem, the functions f and g will always refer to the function defined by the equation:
$$f(x) = \tfrac{1}{2}x^3 - x^2 + 3\pi,$$

and the function whose derivative is graphed below.

$$y = g'(x)$$

(a)   Using any rules that you know for finding derivatives, find an equation for $f'(x)$.

(b)   Let *m* be the function defined by the equation: $m(x) = f(x) + g(x)$. Find the value of $m'(0)$.

(c)   Let *r* be the function defined by the equation: $r(x) = m'(x)$. Find an equation for $r(x)$ that is valid on the interval $0 \leq x \leq 1$.

16.   In this problem the function *f* will always refer to the function defined by the following equation:

$$f(x) = \sqrt{x}.$$

(a)   Use the power rule (or any other short-cut rules that you know) to find an equation for the derivative $f'(x)$.

(b)   Set up the difference quotient that you would use to find an equation for $f'(x)$.

(c)   Simplify your answer to Part (b) as much as possible. When you have finished simplifying, you should no longer have any factors of *h* in the denominator of your difference quotient.

Note: One identity that you might find helpful in Part (c) is:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x+h} - \sqrt{x}\right) \cdot \left(\sqrt{x+h} + \sqrt{x}\right)}{h \cdot \left(\sqrt{x+h} + \sqrt{x}\right)}.$$

(d)   Use your answer to Part (c) to calculate an equation for the derivative $f'(x)$ using the limit definition of the derivative.

17. In this problem, the function g will always refer to the function defined by the following equation:

$$g(x) = \begin{cases} x^2 & ,x < 1 \\ 2x-1 & ,x \geq 1 \end{cases}.$$

In most of the examples or problems that you have investigated, the problem has either been about functions defined in pieces or the limit definition of the derivative but never both. In this problem, you will look at what the value of the derivative is at the boundary where two of the "pieces" of a function meet.

(a) Sketch a graph showing $y = g(x)$ between $x = 0$ and $x = 2$.

(b) Based on the appearance of your graph, what do you think the value of $g'(1)$ should be?

The limit definition of the derivative that you are familiar with mentions only the "overall" limit of the function. As you know, in order for the "overall" limit to exist the left hand limit and the right hand limits must exist. In the next two parts of this question, you will calculate the left and right hand limits of the difference quotient for the function g at the point where $x = 1$.

(c) The appropriate difference quotient for the left hand limit is: $\dfrac{g(1) - g(1-h)}{h}$. Use the equations defining g to calculate this difference quotient, simplify and then take the limit as $h \to 0$.

(d) The appropriate difference quotient for the right hand limit is: $\dfrac{g(1+h) - g(1)}{h}$. Use the equations defining g to calculate this difference quotient, simplify and then take the limit as $h \to 0$.

(e) According to the limits that you have calculated in Parts (c) and (d), does the "overall" limit of the difference quotient exist at $x = 1$? If so, what is the value of $g'(1)$?

18. In this problem, the function $f(x)$ will always refer to the function defined by the equation:

$$f(x) = \frac{1}{x}.$$

(a) Set up the difference quotient of $f(x)$.

(b) Use the difference quotient that you found in Part (a) to find an equation for the derivative of $f(x)$.

(c) By re-writing the equation for $f(x)$, use one of the differentiation rules that you know to find an equation for the derivative of $f(x)$.

(d) Find an equation for the tangent line to the graph of $y = f(x)$ at the point (1, 1).

(e) Sketch a graph showing both $y = f(x)$ and the tangent line whose equation you found in Part (d).

19. Let $P(t)$ represent the total petroleum reserves on Earth at the beginning of year 't.' $P(t)$ is measured in barrels. You may assume that no new petroleum is being made (except in the last question).

(a) What are the units of $\frac{dP}{dt}$?

(b) What is the sign of $\frac{dP}{dt}$ when t = 1999 ? Write a sentence or two explaining why your answer makes sense.

(c) Explain the meaning of the following algebraic expression in practical terms: $\left.\dfrac{dP}{dt}\right|_{t=1980} = R$.

(d) At the Motunui Synthetic Petrol plant in Taranaki, New Zealand, petroleum products are made out of natural gas (see Figures 23.1 and 23.2[1]). The rate at which petroleum is produced is too small to affect the sign of $\frac{dP}{dt}$. What effect does the Motunui plant have on the size of $\frac{dP}{dt}$?

Figures 23.1 and 23.2: Synfuel facility on the North Taranaki coast, New Zealand, and a methanol production unit located in the Synfuel complex.

20. In this problem, the function $f(x)$ will always refer to the inverse sine function, that is:

$$f(x) = \sin^{-1}(x).$$

WARNING: Before starting this problem, make sure that your calculator is set in RADIAN mode. On a TI-83, you can verify that this is the case by pressing the MODE button and ensuring that the word "Radian" is highlighted (see Figure 27.1 below).

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[1] Image source: Weake, H. J. (2000) "Taranaki - A good place to do business." in 2000 New Zealand Petroleum Conference Procedings, Crown Minerals, Ministry of Economic Development, P.O. Box M73, Wellington, New Zealand.
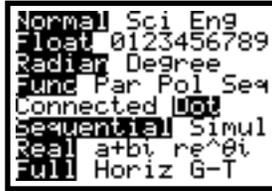
Figure 27.1: Set your calculator to Radian mode.

(a)   Use your calculator to complete the table given below. Use the table to estimate $f'(0.5)$.

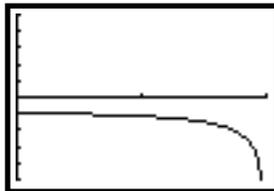| h	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{f(0.5+h)-f(0.5)}{h}$					

(b)   With similar calculations to the ones used to complete the table in Part (a), complete the table shown below.

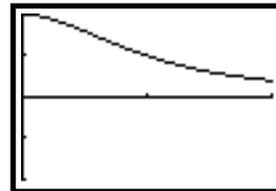| x	0	0.25	0.5	0.75	0.9
$f'(x)$					

(c)   Plot a graph showing $f'(x)$ versus x. Which of the following graphs (see Figure 27.2 below) does your graph resemble the most closely?
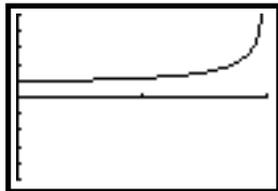
$$y = \frac{1}{\sqrt{1-x^2}} \qquad\qquad y = \frac{-1}{\sqrt{1-x^2}} \qquad\qquad y = \frac{1}{1+x^2}$$

Figure 27.2: Possible graphs for the derivative of $f(x)$.

(d)   Is it possible to find $f'(1)$? Explain your reasoning.

21.   The largest rocket successfully launched was the Saturn V rocket used to carry men to the moon during the late 1960's and early 1970's (see Figure 33.1 below).   The Saturn V consisted of three propulsion stages and a final stage carrying the astronauts and their equipment.
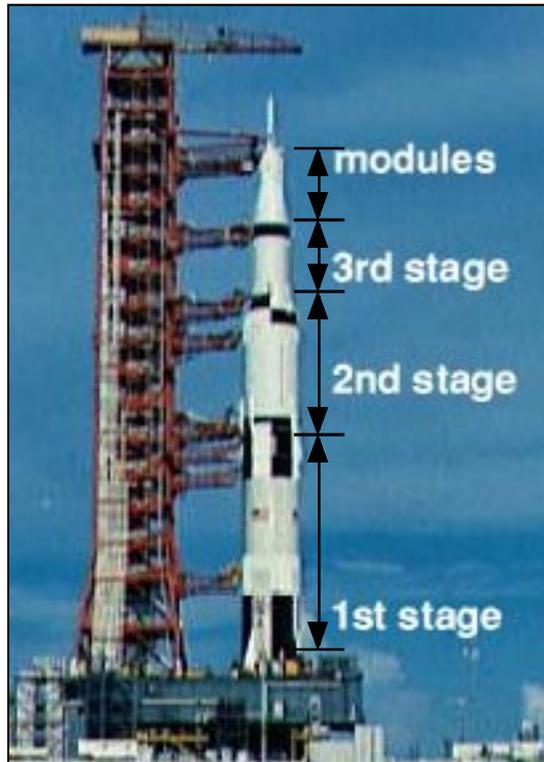

Figure 33.1: The Saturn V launch vehicle.
(The three propulsion stages are indicated.)

Some of the performance data for the Saturn V rocket are given in Table 33.1 below.

Stage	Initial mass* (kg)	Fuel consumption (kg per second)	Exhaust velocity (meters per second)	Total burn time (seconds)
First stage	2286217	13360.24	2496.96	161
Second stage	490778	1158.28	4800.22	390
Third stage	119900	253.2	4800.22	421

* The initial mass includes the masses of the modules and all stages that have not yet fired.

(a)   Find a collection of functions that give the total mass of the Saturn V rocket at all times between ignition of the first stage and the end of the burn of the third stage.

(b)   The velocity of a single-stage rocket at a time T seconds after ignition is given by the equation:

$$v(t) = v_0 + v_E \cdot \ln\left(\frac{M_0}{M(t)}\right)$$

where v_0 is the velocity at the time of ignition, v_E is the exhaust velocity and M_0 is the initial mass at the time of ignition. Find a collection of functions that will give the velocity of the Saturn V rocket at all times between ignition and the end of the burn of the third stage.

(c) When does the Saturn V rocket reach maximum velocity? What is the maximum velocity that the rocket attains?

22. In this problem, the function $f(x)$ is always the function defined in Figure 35.1 below.
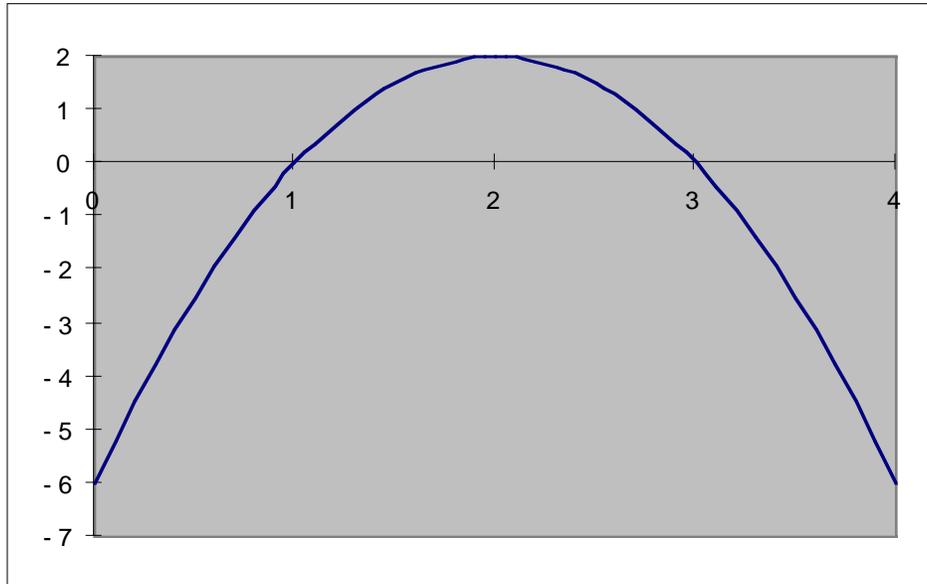

Figure 35.1: Graph defining $f(x)$.

In this problem, the function $g(x)$ is always the function defined by the equation: $g(x) = 10^x$.

Use the functions $f(x)$ and $g(x)$ to calculate the values of the following derivatives:

(a) $f'(2)$.

(b) $k'(2)$ where $k(x) = \frac{f'(x)}{g'(x)}$.

(c) $m'(1)$ where $m(x) = f(x) \cdot g(x)$.

23. A physics student has just finished as experiment to determine how plasma particles move in a magnetic field. After taking many measurements, her computer prints out the *velocity-time* and *acceleration-time* graphs of the plasma particle, as shown below (see Figure 39.1 and 39.2). The computer prints out the very first *position* of the plasma particle (it prints $s(0) = 2$), but then crashes destroying all the data. The physics student is devastated, and comes to you for support. Help her out by using the information given to accurately sketch a graph of *position versus time*.
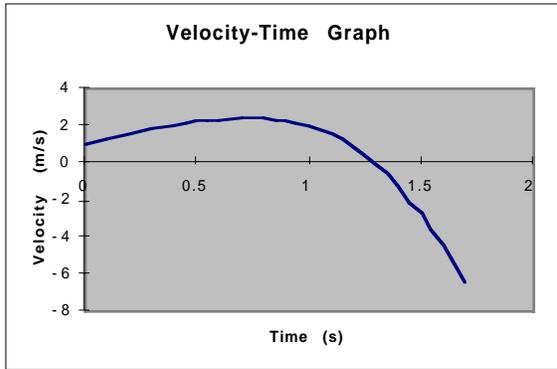
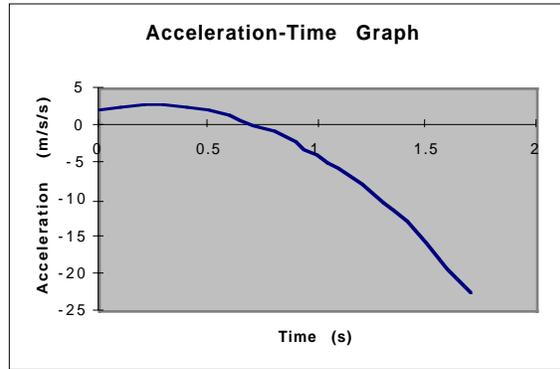Figure 39.1: Velocity-time graph for plasma particle.

Figure 39.2: Acceleration-time graph for plasma particle.

24. In this problem, the function $f(x)$ always refers to:

$$f(x) = x^3 - 5x + 1.$$

(a) Find the critical point(s) of f.

(b) Where is f increasing and where is f decreasing?

(c) Where are the inflection points of f ?

(d) Where is f concave up and where is f concave down?

(e) What is the global maximum of f on the interval [0, 5] ?

(f) What is the global minimum of f on the interval [0, 5] ?

25. The graph of the *derivative*, $F'(x)$, of a function $F(x)$ is shown in Figure 42.1 below.
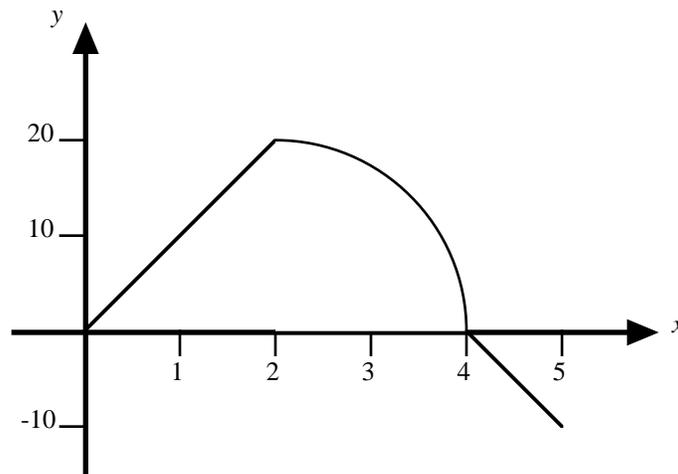

Figure 42.1: Graph of the *derivative* of the function $F(x)$.

(a) Suppose that $F(1) = 100$. What is the global maximum of F on the interval $[0, 5]$? What are the coordinates of the global maximum?

(b) Explain how you know that the point you found in Part (a) is a *local* maximum.

(c) Explain how you know that the point you found in Part (a) is a *global* maximum.

26. The relationship between the independent variable x and the dependent variable, y, is shown in Figure 13.1 below.
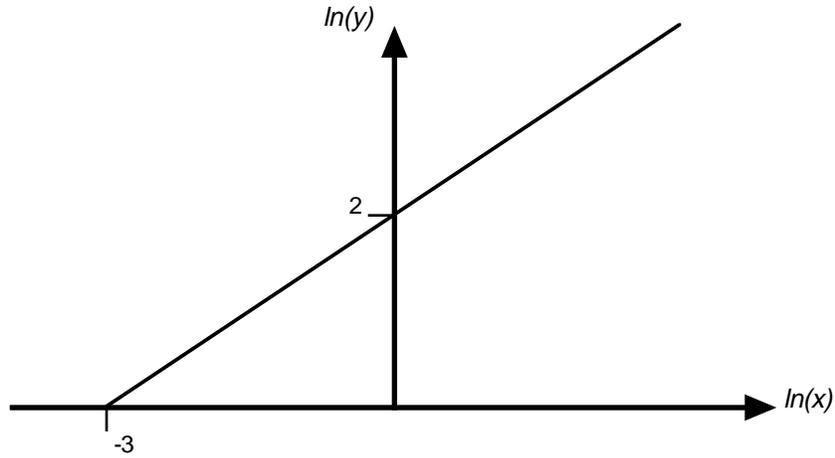

Figure 13.1

(a) What sort of function relates $ln(y)$ to $ln(x)$? Find the equation for this function.

(b) What sort of function relates y and x ? Find the equation for this function.

(c) What value of x gives $y = e^2$?

27. After an unsuccessful campaign to market oil in long, rolled-up tubes (called "Coil 'O' Oil"), a company resorts to selling their oil in more conventional, cylindrical containers. The company plans to sell 500 ml cylindrical cans of oil, but the packaging material for the ends of the can costs twice as much as the material for the sides. If the company only cares about how much the packaging costs, what should the dimensions of their oil can be?

28. The great white shark (Charcharodon carcharius) is a magnificent animal that is perfectly adapted to its environment (see Figure 70.1). This creature is feared and maligned by many people because of the image of the shark as a mindless, savage, sadistic killer from movies like "Jaws." In reality, more people are killed each year by bees than by great white sharks.

Figure 70.1: A great white shark
(*Carcharodon carcharias*).

It was long thought that the habitat of the great white included only North America, southern Africa, Australia, New Zealand and Japan. However, evidence has been slowly accumulating to suggest that great whites may also live in the Mediterranean sea.

In 1984, a German fisherman caught a great white off the coast of an Italian beach resort. Concerned about the possible impact on tourism, and to prevent a massacre of sharks, the owners of the beach resort hushed the incident up. In the subsequent years, the owners kept an eye on shark numbers, collecting the statistics shown in Table 1.

Number of Sharks (P)	13	16	21
Relative Growth Rate ($\frac{1}{P}\frac{dP}{dt}$)	1.5	1.4	1.2

Table 1

(**a**)   Use the data in Table 1 to find a differential equation that describes the change in the shark population over time.

(**b**)   Experts from the University of Hawaii estimate that when fifty or more sharks are living along a stretch of coastline, even tourists will start to notice them. Prior to 1984, no tourists at the Italian resort had spotted a shark. Is there a danger of tourists spotting sharks in the future? Explain your reasoning with the aid of a slope field.

Answers

These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.

1.(a) The limit is equal to 5.
1.(b) The limit does not exist as the height of the graph $y = f(x)$ does not settle down to any consistent value. Instead, it just oscillates back and forth.
1.(c) The limit does not exist as the left and right hand limits are not equal.
1.(d) The limit does not exist as the left and right hand limits are not equal.
1.(e) The limit is equal to 1.
1.(f) The limit does not exist as the left and right hand limits are not equal.
1.(g) The limit is equal to 1.
1.(h) The limit is equal to 2.
1.(I) The limit is equal to $+\infty$.

2.(a) The average speed is: $\dfrac{s(10) - s(0)}{10 - 0} = 26.4$ feet per second.

2.(b) The speedometer reading will be equal to $s'(10) = 52.8$ feet per second. There are 5280 feet in a mile and 3600 seconds in an hour, so the speedometer will read 52.8*3600/5280 ≈ 36 miles per hour.

**2.(c)** The speedometer reading will be equal to $s'(30) = 105.6$ feet per second. There are 5280 feet in a mile and 3600 seconds in an hour, so the speedometer will read 105.6*3600/5280 ≈ 72 miles per hour.

2.(d) The average speed over the interval [10, 10 + k] will be: $\dfrac{2.64 \cdot (10 + k)^2 - 2.64 \cdot 10^2}{k}$.

2.(e) The limit is equal to 52.8 (which also happens to be the derivative of $s(t)$ when $t = 10$).

3.(a) The line is shown on the diagram below.
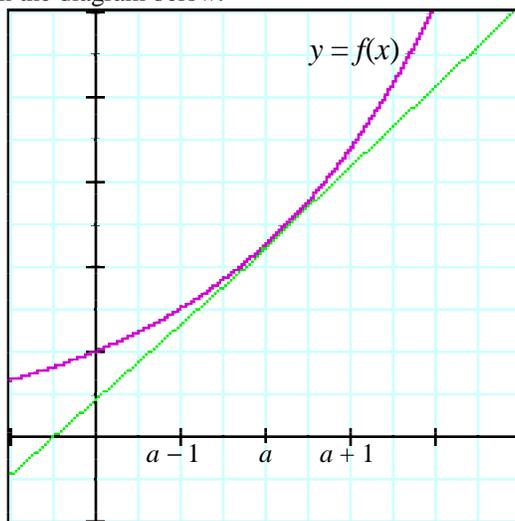
3.(b) The equation for the line is: $y = f'(a) \cdot x - f'(a) \cdot a + f(a)$.
3.(c) When $x = a - 1$, $y = -f'(a) + f(a)$.
    When $x = a + 1$, $y = f'(a) + f(a)$.
3.(d) The statement means that when time is increased from 7 hours to 8 hours, the number of bacteria in the colony will increase by about 14 bacteria.

4.(a) The graph of $d(t)$ is shown in the diagram below.

4.(b) Factoring the equation for $d(t)$ gives: $d(t) = -16t(t - 3)$. This equation is equal to zero when $t = 0$ and when $t = 3$. The first time that $d(t) = 0$ is when $t = 0$. This is probably when the tomato was launched into the air. The second time when $d(t) = 0$ is when $t = 3$. This is probably when the tomato landed on the ground.

4.(c) The time when height, $d(t)$, was maximum can be found by locating the time when $d'(t) = 0$. Differentiating the equation for $d(t)$ gives:

$$d'(t) = -32t + 48.$$

Setting this derivative equal to zero and solving for t gives that the tomato reaches its maximum height when $t = 1.5$.

4.(d) The value of the maximum height will be given by $d(1.5) = 36$ feet.

4.(e) The equation for $d(t)$ in vertex form is: $d(t) = -16(t - 1.5)^2 + 36$.

4.(f) The values of t and $d(t)$ at maximum height are the x- and y-coordinates of the vertex of the graph shown in Part (a). If the coordinates of the vertex of a quadratic equation are (h, k) then the equation for the quadratic in vertex form is: $q(x) = a \cdot (x - h)^2 + k$. So, since you know $h = 1.5$ and $k = 36$ from Parts (d) and (e), you could have obtained the vertex form by just plugging these numbers into $q(x) = a \cdot (x - h)^2 + k$. The value of a could be determined by plugging one of the points from Part (b) into this equation and solving for a.

5.(a) $y = j(x)$.

5.(b) $y = g(x)$.

5.(c) $y = h(x)$.

5.(d) $y = f(x)$.

5.(e) This statement does not match any of the functions.

6.(a) Using the methods that you learned in the Biomedical Data lab, note that the zeros of f lie at $x = -5$, $x = -1$ and $x = 3$. These are all zeros of multiplicity 1 so the equation for $f(x)$ will resemble:

$$f(x) = k \cdot (x + 5) \cdot (x + 1) \cdot (x - 3),$$

where k is a constant. To determine k, note that the point $(0, -0.5)$ lies on the graph. Plugginf $x = 0$ and $y = -5$ into the equation gives $k = 1/30$. Hence:

$$f(x) = \tfrac{1}{30} \cdot (x + 5) \cdot (x + 1) \cdot (x - 3).$$

6.(b) The intersection points occur when $x = 2$ and $x = 4$. Using the equation for $f(x)$ to find the y-coordinates of these points gives that the points of intersection are $(2, -0.7)$ and $(4, 1.5)$.

6.(c) The equation for L is the equation of the secant line that goes through the points $(2, -0.7)$ and $(4, 1.5)$. The equation of this line is: $y = 1.1x - 2.9$.

7.(a) The graph of $y = f(x)$ is shown in the diagram below.
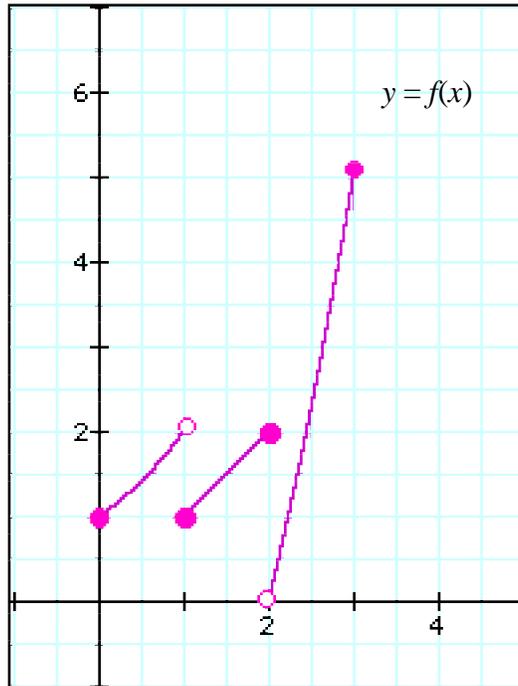
7.(b) **(I)** 1 **(II)** 2 **(III)** 0 **(IV)** 2.

7.(c) No it is not always the case. The diagram below shows the graph of a function g. The value of $g(1)$ is 2, but the left and right hand limits of g as x approaches 1 are both equal to -1. Therefore, the value of the function does not have to equal either the right or the left hand limit.
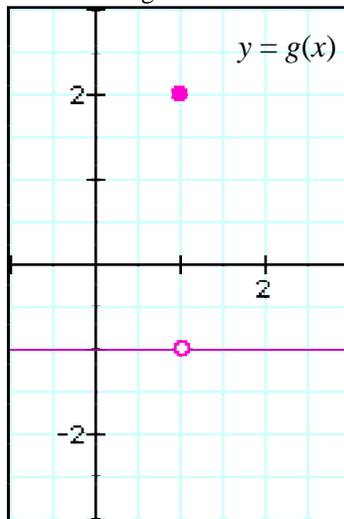
8. Of the four available graphs, graph (d) is closest to satisfying the requirements for f. Neither (b) nor (c) have the derivative equal to zero when $x = 0$. Graph (a) is increasing when $x = -1$ and decreasing when $x = +1$, which is the precise opposite of what is wanted for f.

9.(a) Let T = age of the sample in years and $M(T)$ = amount of carbon-14 remaining in the sample (measured in micrograms). Then:

$$M(T) = 0.0001 \cdot \left(\frac{1}{2}\right)^{\frac{T}{5730}}.$$

9.(b) The age of the matter will be 9240.41 years.

9.(c) Using the figure of 0.0000372 gives and age of 8174.56. This will be 1065.85 years too young.

9.(d) The correct age will be approximately 2134 years old.

10.(a) Let T = number of years since 1990, and $P(T)$ = population of Hungary in millions of people. Then the equation for the Hungarian population is: $P(T) = 10.8 \cdot (0.998)^T$.

10.(b) The derivative of $P(T)$ is given by: $P'(T) = 10.8 \cdot (0.998)^T \cdot \ln(0.998)$. Evaluating this when $T = 10$ gives: $P'(10) = -0.021193$. The units of $P'(T)$ are millions of people per year.

10.(c) A practical interpretation of $P'(10) = -0.021193$ is that between $T = 10$ (i.e. the beginning of the year 2000) and $T = 11$ (i.e. the beginning of the year 2001) the population of Hungary would decrease by approximately 21193 people.

10.(d) The inverse of $P(T)$ is: $T(P) = \dfrac{\ln(P) - \ln(10.8)}{\ln(0.998)}$. (Note that you could also have used the common logarithm function here - in which case your equation would involve "log" rather than "ln.")

10.(e) The derivative of the inverse is given by: $T'(P) = \dfrac{1}{\ln(0.998) \cdot P}$.

10.(f) When $P = 3.4$, the derivative is: $T'(3.4) = -146.9$.

10.(g) One practical interpretation of this number is that it will take approximately 146.9 years for the population of Hungary to drop from 3.4 to 2.4 million.

11.(a) The derivative of f is: $f'(x) = \dfrac{e^{-x}}{\left(1 + e^{-x}\right)^2}$. This quantity is positive, so the original function f is increasing.

11.(b) As f is always increasing, it's graph is always going up when read from left to right. This means that f will pass the horizontal line test. The argument for this goes as follows: suppose that f didn't pass the horizontal line test. Then there would have to be two points of the graph $y = f(x)$ that had the same height. Therefore, the graph of f must have decreased when read from left to right in order to come down to that height a second time. However, the one thing that you definitely know about f is that it is always increasing, so it can't ever have decreased. That means that the assumption that the graph $y = f(x)$ doesn't pass the horizontal line test must be wrong. A formula for the inverse of f is given by:

$$f^{-1}(x) = \ln\left(\frac{1}{1-x}\right).$$

12. If the hotel chain takes only 300 chairs then the total revenue will be $27,000. If the hotel chain takes x chairs (where x is between 300 and 400) then the revenue is given by:

$$R(x) = x \cdot \left(90 - 0.25 \cdot (x - 300)\right).$$

Differentiating this function and setting the derivative equal to zero gives $x = 330$. Checking the sign of the derivative on each side of $x = 330$ confirms that this is a local maximum. When $x = 330$, the revenue will be $R(330) = \$27,225$. The one other point to check is the end-point $x = 400$. The revenue when $x = 400$ is: $R(400) = \$26,000$. Therefore, the greatest revenue that the furniture company can collect is $27,225 and the lowest is $26,000.

13.(a) Cup A - Graph 3.   Cup B - Graph 4.   Cup C - Graph 1.   Cup D - Graph 2.

13.(b) $\left(d^{-1}\right)'(10) = 0.7$.

14.(a) As $k(x)$ is a product of two functions, you use the product rule to differentiate it. This gives:
$$k'(x) = \left(4x^3 - 1\right)\cdot\left(\tfrac{1}{2}x^2 + 2\right) + (x)\cdot\left(x^4 - x - 1\right).$$

14.(b) As $m(x)$ is a quotient of two functions, you use the quotient rule to differentiate it. Doing this gives $m'(2) = 8$.

14.(c) You have enough information to work out $m(2)$, as you are told that $h(2) = 1$ and you can use the equation for $g(x)$ to work out that $g(2) = 4$. Therefore, $m(2) = 4$. The derivative $m'(2)$ is approximately how much the function m will change when x is increased from $x = 2$ to $x = 3$. Since $m'(2) = 8$, the function will increase by approximately 8 when x is increased from $x = 2$ to $x = 3$. Therefore, the value of $m(3) \approx 4 + 8 = 12$.

15.(a) $f'(x) = \tfrac{3}{2}x^2 - 2x$.

15.(b) When $x = 0$, the graph of $y = g'(x)$ has height 3. This means that the derivative of g will be equal to three at that point. In symbols, that means that: $g'(0) = 3$. The derivative of $m(x)$ at $x = 0$ is: $m'(0) = f'(0) + g'(0) = 0 + 3 = 3$.

15.(c) $r(x) = m'(x) = f'(x) + g'(x) = \tfrac{3}{2}x^2 - 2x + 3$.

16.(a) $f'(x) = (1/2)x^{-1/2}$.

16.(b) The difference quotient for $f(x)$ will be:
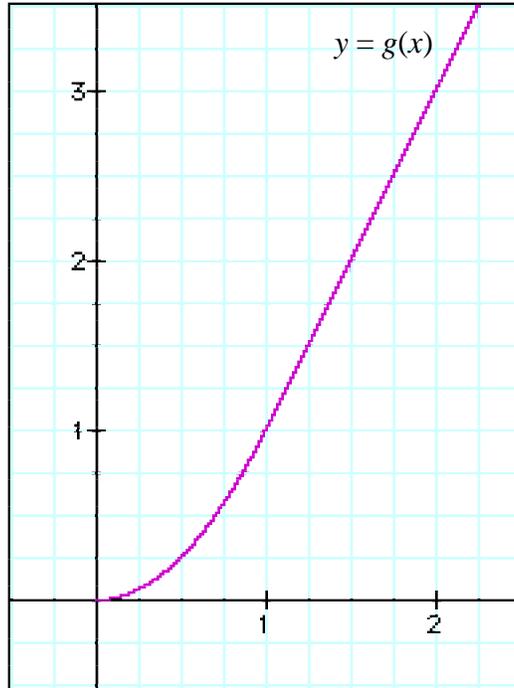$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

16.(c) Simplifying the difference quotient as much as possible gives:
$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\cdot\left(\sqrt{x+h} + \sqrt{x}\right)}{h\cdot\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{x+h-x}{h\cdot\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

16.(d) Taking the limit of the simplified difference quotient as $h \to 0$ gives:
$$f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

17.(a) The graph of $y = g(x)$ is shown in the diagram below.

$y = g(x)$

17.(b) When $x = 1$, the graph of $y = g(x)$ appears to resemble a straight line. The slope of this line is 2, so I would expect the value of $g'(1)$ to be 2.

17.(c) The left hand difference quotient is:

$$\frac{g(1) - g(1-h)}{h} = \frac{1 - (1-h)^2}{h} = \frac{2h - h^2}{h} = 2 - h.$$

Taking the limit of this as $h \to 0$ gives a limit of 2.

17.(d) The right hand difference quotient is:

$$\frac{g(1+h) - g(1)}{h} = \frac{2 + 2h - 1 - 1}{h} = \frac{2h}{h} = 2.$$

Taking the limit of this as $h \to 0$ gives a limit of 2.

17.(e) As the left hand and right hand limits are both equal to 2, the overall limit exists and is equal to 2.

18.(a) The difference quotient is shown below. (I have simplified the difference quotient a little in anticipation of Part (b) of this problem.)

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{x - (x+h)}{h \cdot x \cdot (x+h)} = \frac{-1}{x \cdot (x+h)}$$

18.(b) The derivative is the limiting value of the difference quotient as $h \to 0$. Using the simplified difference quotient obtained in Part (a) above gives: $f'(x) = -1/x^2$.

18.(c) Re-writing the equation for $f(x)$ gives: $f(x) = x^{-1}$. Using the power rule for derivatives: $f'(x) = -x^{-2}$.

18.(d) Substituting $x = 1$ into the equation for the derivative gives the slope of the tangent line. So, the slope of the tangent line is -1. Substituting $m = -1$, $x = 1$ and $y = 1$ into the equation for a linear function gives the intercept, b, of the equation for the tangent line: $b = 2$. So, the equation for the tangent line is:

$$y = -x + 2.$$

18.(e) A plot showing the graph of $y = f(x)$ and the graph of $y = -x + 2$ is shown below.
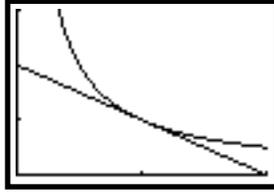
19.(a) The units of the derivative will be barrels of oil per year.
19.(b) The sign of the derivative (when $t = 1999$) will be negative. This is because more petroleum was consumed in 1999 than was produced, so the world petroleum resources were reduced.
19.(c) The statement means that between 1980 and 1981, the amount of petroleum in the world changed by approximately R barrels. Since R is negative, this means that the amount of petroleum in the world was about $|R|$ barrels less in 1981 as compared to 1980.
19.(d) The output of the Motunui plant will make the derivative slightly less negative. The output of the Motunui plant will not be sufficient to actually change the derivative from negative to positive, but it will make the derivative slightly less negative.

**20.(a)** The completed table is given below.

H	0.01	0.001	0.0001	0.00001	0.000001
$\frac{f(0.5+h)-f(0.5)}{h}$	1.1586	1.15508	1.154739	1.154704	1.1547009

The value of the derivative, $f'(0.5)$, is the limiting value of the difference quotient from the table as h approaches zero. This limiting value appears to be approximately equal to 1.1547.
20.(b) The completed table is shown below.

x	0	0.25	0.5	0.75	0.9
$f'(x)$	1	1.032795	1.1547	1.511858	2.2942

20.(c) A plot a graph showing $f'(x)$ versus x is given below. (This is simply a plot of the data points from the table in Part (b)).
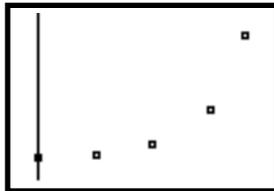
20.(d) No it is not possible to calculate $f'(1)$. The domain of the inverse sine function is all numbers between $x = -1$ and $x = 1$ (inclusive). Therefore, you would need to modify the difference quotient to something like:

$$\frac{f(1)-f(1-h)}{h} \ .$$

If you make a table to try to find the limiting value of this difference quotient as h approaches zero, then you will discover that the difference quotient does not settle down to a limiting value, but instead grows larger and larger the closer to zero that h is made.

21.(a) The collection of functions is given in the table below. All masses are in kilograms and all times are in seconds.

Time interval (seconds)	Mass (kg)
0 < T ≤ 161	2286217 − 13360.24*T
161 < T ≤ 551	490778 − 1158.24*(T − 161)
551 < T ≤ 972	119900 − 253.2*(T − 551)

21.(b) The collection of functions is given in the table below. All velocities are in meters per second and all times are in seconds.

Time interval (seconds)	Velocity (m/s)
0 < T ≤ 161	$2496.96 \cdot \ln\left(\frac{2286217}{2286217-13360.24*T}\right)$
161 < T ≤ 551	$7060.81 + 4800.22 \cdot \ln\left(\frac{490778}{490778-1158.28*(T-161)}\right)$
551 < T ≤ 972	$19211.03 + 4800.22 \cdot \ln\left(\frac{119900}{119900-253.2*(T-551)}\right)$

21.(c) If you draw a graph of the velocity function, you will see that it is always increasing. Therefore, the maximum value of velocity will be reached at the highest value of T that is allowed. In this case, that is $T = 972$. When $T = 972$ the velocity of the Saturn V rocket is about 29734.9 m/s.

22.(a) From the graph of $y = f(x)$, $f'(2) = 0$.
**22.(b)** Using the quotient rule for derivatives,

$$k'(2) = \frac{f'(2) \cdot g(2) - f(2) \cdot g'(2)}{[g(2)]^2} = \frac{0*10^2 - 2*\ln(10)*10^2}{[10^2]^2} = -0.04605.$$

22.(c) In this part of the problem, you have to come up with a value for $f'(1)$. I did this by drawing a tangent line to $x = 1$ on the graph of $y = f(x)$ and then estimating the slope of that tangent line. Using the product rule for derivatives: $m'(1) = f'(1)*g(1) + f(1)*g'(1) = 4*10 + 0*\ln(10)*10 = 40$.

23. A possible position-time graph is shown below. There are many possible answers to this question. The essential features that a correct answer to this problem should have are:

- Graph is increasing between t=0 and t=1.25.
- Graph has a maximum around t=1.25.
- Graph is decreasing after t=1.25.

- Graph is concave up between t=0 and t=0.75.
- Graph has a point of inflection around t=0.75.
- Graph is concave down after t=0.75.

Position-Time Graph
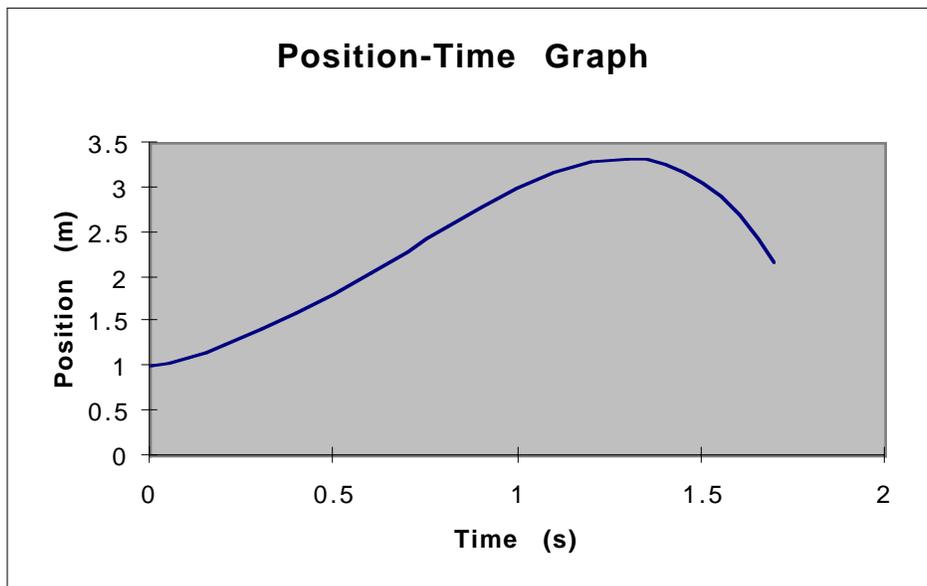
24.(a) The critical points of the function are the points where $f'(x) = 0$. The derivative of $f(x)$ is given by:
$$f'(x) = 3x^2 - 5.$$
Setting this equation equal to zero and solving for x gives: $x = \pm\sqrt{(5/3)}$. Evaluating the function $f(x)$ at these x-values gives the coordinates of the critical points:
$$(-(5/3)^{1/2}, 3.3) \text{ and } ((5/3)^{1/2}, -3.3).$$

24.(b) The function is increasing where the derivative is positive and decreasing where the derivative is negative. The function $f(x)$ is:

- Increasing on intervals: $-\infty < x < -(5/3)^{1/2}$ and $(5/3)^{1/2} < x < \infty$.
- Decreasing on interval: $-(5/3)^{1/2} < x < (5/3)^{1/2}$.

24.(c) Points of inflection usually occur where the concavity of the original function changes. It is always a good idea to check the graph of the function to make sure that there really is a point of inflection where you think there might be one. Checking the graph of $y = f(x)$ (see below) shows that the concavity of $f(x)$ appears to change at $x=0$, so there is a point of inflection at $x=0$.
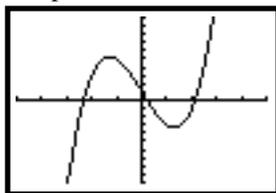
24.(d) The function is concave up where the derivative is increasing and concave down where the derivative is decreasing. The function is then:

- Concave up on interval: $-\infty < x < 0$
- Concave down on interval: $0 < x < \infty$.

24.(e) The global minimum is the highest value that the function can have when x is restricted to the values $0 \le x \le 5$. Inspecting the graph of $y = f(x)$ (see below) shows that the global maximum is: (5, 101).
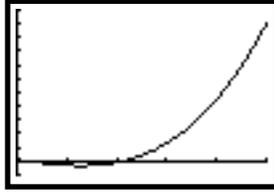
24.(f) By inspecting the graph, the global minimum is located at the minimum that was detected in Part (a). Therefore, the global minimum is: $(-(5/3)^{1/2}, -3.3)$

25.(a) The global maximum occurs at $x = 4$. The coordinates of the global maximum are approximately $(4, 142)$.

25.(b) $x = 4$ is a local maximum because the derivative is positive immediately to the left of $x = 4$, and negative immediately to the right of $x = 4$.

25.(c) There are no other critical points (that aren't end points) in the interval $[0, 5]$. So, the only other places that a global maximum could occur are at $x = 0$ and $x = 5$. The derivative is positive between $x = 0$ and $x = 4$, so $F(4) > F(0)$. The derivative is negative between $x = 4$ and $x = 5$, so $F(4) > F(5)$.

26.(a) It is a linear Function. $ln(y) = (2/3)*ln(x) + 2$.
26.(b) It is a power Function. $y = (7.4)*x^{(2/3)}$
26.(c) $x = 1$.

27. Radius = 3.4139 cm. Height = 13.6556 cm.

28.(a) The equation is: $\dfrac{dP}{dt} = P \cdot (1.9959 - 0.03775 \cdot P)$.

28.(b) The slope field is not shown here, but the pattern that it portrays is something like this:

- There is a row of horizontal line segments when $P = 0$.
- There is a row of horizontal line segments when P is approximately 53.
- Above $P = 53$, the line segments slope downwards.
- Between $P = 0$ and $P = 53$ the line segments slope upwards.

When you use the slope field to sketch a graph showing the number of sharks as a function of time, the graph eventually gets above 50 sharks (it levels off near 53 sharks). This means that as time goes on, the growing shark population may take a sizable bite out of tourism revenues in Italy, although the sharks themselves are very unlikely to take a bite out of any actual tourists.