Practice Problems: Final Exam

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Friday, January 18.

2. The test will include ten to twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test.

5. The specific topics that will be tested are:

- Representations of data (graphs, tables, equations, verbal descriptions) and the concept of a function.
- Interpreting graphs of functions.
- Average rate of change and relationship between rate of change and concavity.
- Properties of linear functions.
- Modeling data using linear functions.
- Exponential growth.
- Modeling data using exponential functions.
- Power functions.
- Modeling data using power functions.
- Vertical and horizontal asymptotes.
- Interpreting and using function notation.
- Arithmetic combinations and compositions of functions.
- Transformations of functions from the algebraic and graphical points of view.
- Inverse functions.
- Polynomial functions.
- Rational functions.
- Quadratic functions
- Functions defined in pieces
- Left and right hand limits
- Calculating limits of functions
- Compound interest and the growth of investments
- Average and instantaneous rates of change
- The derivative function
- Calculating derivatives using the definition
- Graphical and verbal interpretations of the derivative
- Calculating derivatives using the short-cut rules.
- Polynomial, rational functions and their graphs
- Solving exponential equations using logarithms
- Derivatives of exponential and logarithmic functions
- The number ‘e,’ and derivative of $y = e^x$
- Locating and classifying the maximum, minimum values of a function.
- Modeling situations with functions and optimization.
- The second derivative and concavity.
- Related rates.
- Implicit differentiation.

- Slope fields and solutions of differential equations.
- Euler's method for solving initial value problems.

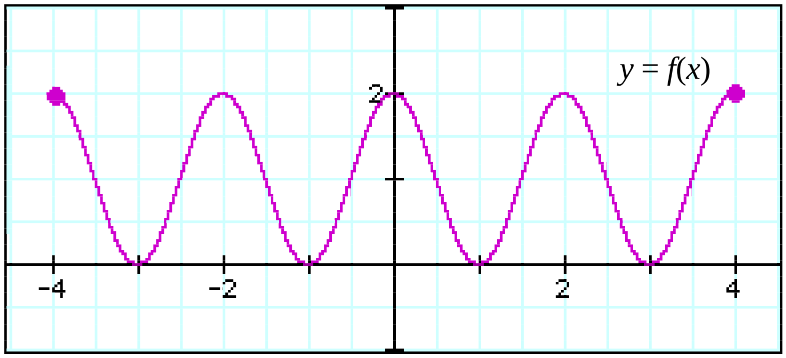
6. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

7. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

8. Good sources of help:

- Section leaders' office hours (posted on Xa web site).
- Math Question Center (during the reading period).
- Course-wide review on Wednesday 1/16 from 7:00-9:00 p.m. and Thursday 1/17 from 7:00-9:00 p.m. both in Science Center 309.

1 . In this problem, the function $f(x)$ will always refer to the function defined by the graph given below. The domain of the function $f(x)$ is the interval $[-4, 4]$. Note that this interval does include the end-points $x = -4$ and $x = 4$.



In this problem, the function $g(x)$ will always refer to the function defined by the equation:

$$g(x) = [f(x)]^2 - 2\cdot f(x).$$

(a) What is the domain of the function $g(x)$?

(b) Find the x-coordinates of the point(s) where the derivative of $f(x)$ is equal to zero.

(c) Express the derivative of $g(x)$, $g'(x)$, in terms of $f(x)$ and the derivative of $f(x)$.

(d) Find the x-coordinates of all points where the derivative of $g(x)$ is equal to zero.

(e) For each of the points that you found in Part (d), determine whether the point is a local maximum, local minimum or neither.

2 . Wild rabbits are the most serious animal pest on the island continent of Australia and the second most serious pest (after the marsupial possum) in the island nation of New Zealand. In both countries wild rabbits originally introduced from Europe have been responsible for extensive ecological damage[1]. The annual cost of efforts to control the rabbit population and loss of agricultural productivity is estimated to be in the region of $600 million per year[2].

Given that rabbits have created serious economic and ecological problems for Australia and New Zealand, it may come as a surprise that rabbits were intentionally released in both countries by European settlers in the 1800's. The introduction of rabbits to Australia is often traced to the importation and release of 24 rabbits by Thomas Austin in 1859[3] on his property in southern Victoria. In 1866, there were at least 14,253 rabbits on Mr. Austin's property and by 1869 this population had grown to at least 2,033,000 rabbits. These numbers are summarized in the table below.

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[1] Source: Brian D. Cooke. The effects of rabbit grazing on regeneration of sheoaks, *Allocasuarina verticillata*, and salt-water ti-trees, *Meloleuca halmaturorum*, in the Coorong National Park. *Australian Journal of Ecology*, **13**(1): 11-20, 1987.

[2] Source: CSIRO, Australia. Environmental damage by wild rabbits. *CSIRO Media Release*, August 17, 1996.

[3] Source: E.C. Rolls. *They All Ran Wild.* Sydney, Australia: Angus and Robertson Publishers, Inc., 1969

Year	Number of years since introduction	Number of rabbits
1859	0	24
1866	7	14,253
1869	10	2,033,000

(a) Based on the figures given in the table above, is the number of rabbits a linear function of the number of years since introduction? Provide evidence for your answer.

(b) Based on the figures given in the table, is the number of rabbits an exponential function of the number of years since introduction? Provide evidence for your answer.

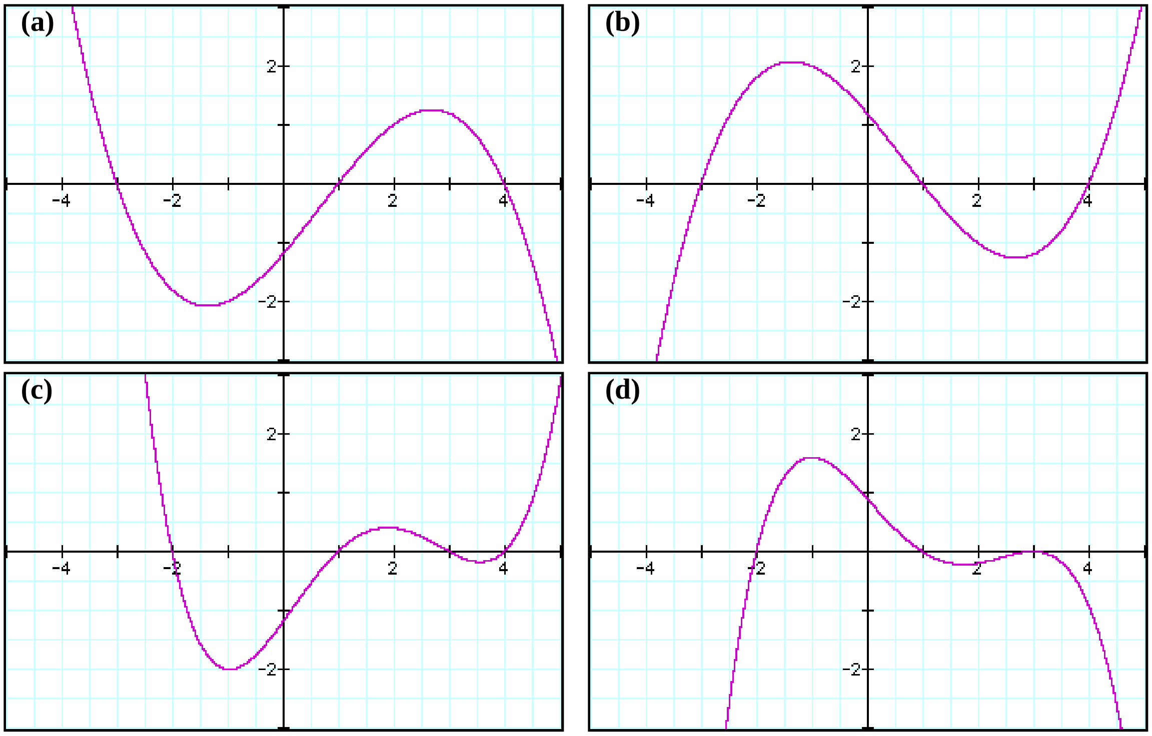
(c) Based on the figures given in the table, is the number of rabbits a power function of the number of years since introduction? Provide evidence for your answer.

(d) Which of these three functions (linear, exponential or power) does the best job in representing the number of rabbits as a function of the number of years since introduction? Find an equation for this function.

(e) Use your function to predict the number of rabbits that are currently living on Mr. Austin's property 143 years after they were first introduced.

(f) A wild rabbit usually has a mass of a few kilograms. Do you think that the estimate from Part (e) is reasonable or not? For comparison, the mass of the sun is about $2*10^{30}$ kilograms. Sketch more realistic graph showing the number of rabbits as a function of time.

3 . Find an equation for each of the polynomial functions shown below. Normally, your last step in such a calculation would be to determine the constant of proportionality, k, for each function. Instead of doing that here, simply determine whether the constant should be positive or negative.



4 . In this problem the function $f(x)$ is a function whose derivative is always defined. The only two pieces of information that you can assume about $f(x)$ are given below:

- $f(1) = 1$
- $f'(1) = 1$

In this problem, the function $g(x)$ will always refer to the function defined by the equation given below:

$$g(x) = \frac{f(x)}{x}.$$

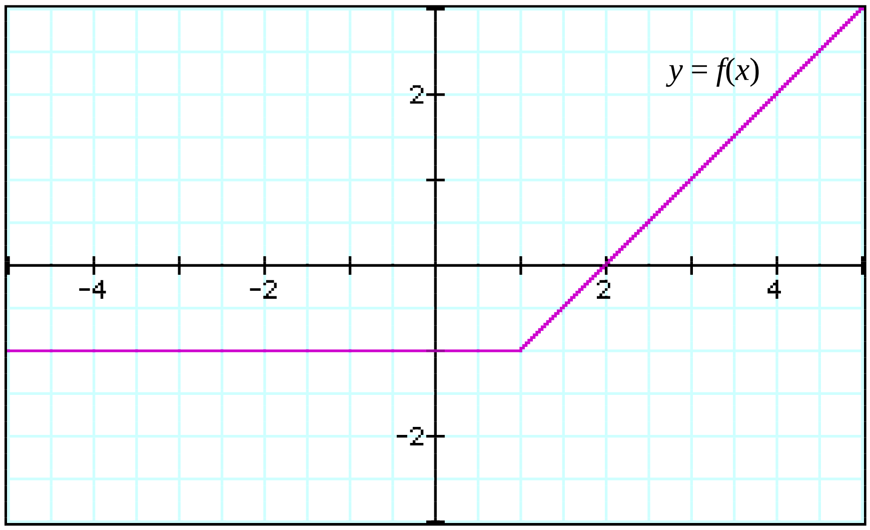
(a) Set up the difference quotient that you need to evaluate in order to find $g'(1)$ using the limit definition of the derivative.

(b) Simplify the difference quotient from Part (a) as much as you possibly can.

(c) Take the limit as $h \to 0$ to calculate $g'(1)$.

(d) Use the quotient rule to confirm your answer to Part (c).

5 . In this problem, $f(x)$ will always refer to the function defined by the graph shown below.



In this problem, the function $g(x)$ will always refer to the function defined by the equation:

$$g(x) = f(x) \cdot e^x.$$

(a) Write down an equation for the function $f(x)$. (Hint: don't try to write down a single formula. Instead break $f(x)$ up into pieces.)

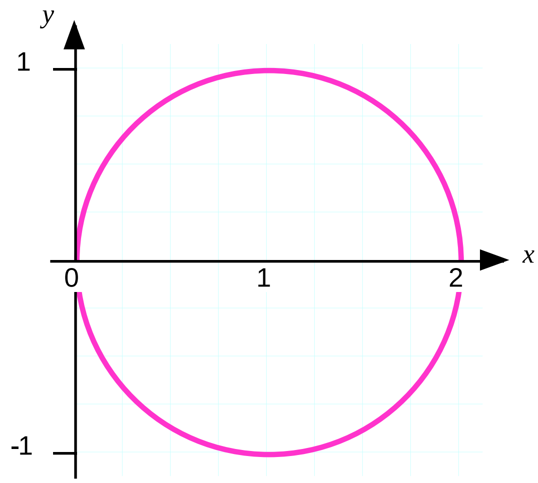
(b) What is the domain of the derivative $g'(x)$?

(c) Write down an equation for the derivative $g'(x)$. (Hint: don't try to write down a single formula, instead break $f(x)$ up into pieces.)

(d) Write down an equation for the second derivative $g''(x)$. (Hint: don't try to write down a single formula. Instead break $f(x)$ up into pieces.)

(e) Find the intervals over which $g(x)$ is concave up and the intervals over which $g(x)$ is concave down.

6 . The diagram given below shows a curve in the *xy*- plane.



(a) The *x*- and *y*-coordinates of the points that lie on this curve satisfy the equation:

$$y^2 + (x-1)^2 = 1.$$

Is it possible to find a function $f(x)$ so that the curve shown above is the graph of $y = f(x)$? If so, find an equation for $f(x)$. If not, explain why not.

(b) Find an equation for the derivative of *y* with respect to *x* - that is, an equation for $\frac{dy}{dx}$.

(c) The portion of the curve that lies above the *x*-axis can be represented by the equation:

$$y = \sqrt{1-(x-1)^2}\,.$$

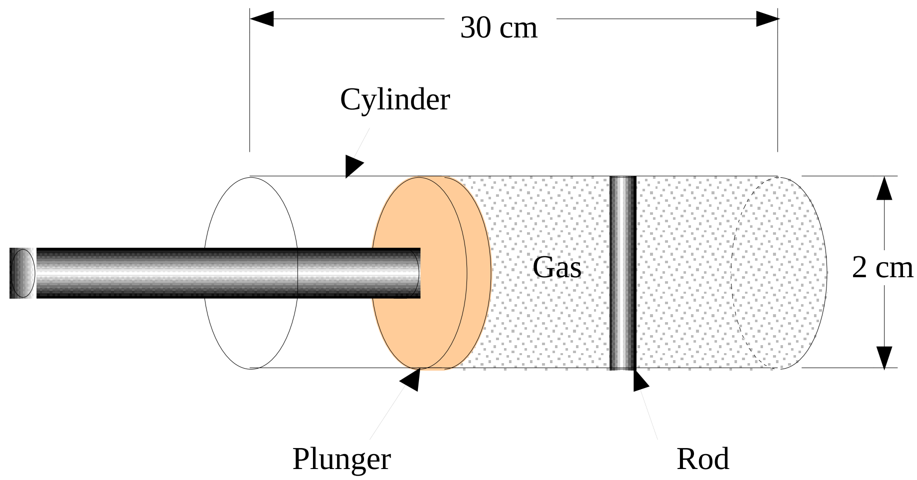
Use this explicit equation for *y* as a function of *x* to find an equation for $\frac{dy}{dx}$.

(d) In Parts (b) and (c) of this problem, you found an equation for $\frac{dy}{dx}$ in two different ways. Chances are, the two equations that you found for $\frac{dy}{dx}$ look quite different as well. Are these two equations really the same or are they different in some important way?

7 . During an experiment in an introductory chemistry class, a gas undergoes adiabatic compression (meaning that no heat is added or removed) in a glass tube 30 cm long and 2 cm in diameter. As the gas is compressed, a student records the following data for pressure and volume :

Pressure (atmospheres)	1.0	1.1	1.3	1.5	2.0	2.5
Volume (cm^3)	94	86	80	71	58	49

The lab manual says that (as a precaution) a metal rod should be placed in the cylinder so that it cannot be over-compressed (and shatter).



Note that the volume of a cylinder is $\pi r^2 l$, where r is the radius, and l is the length of the portion of the cylinder that is filled with gas.

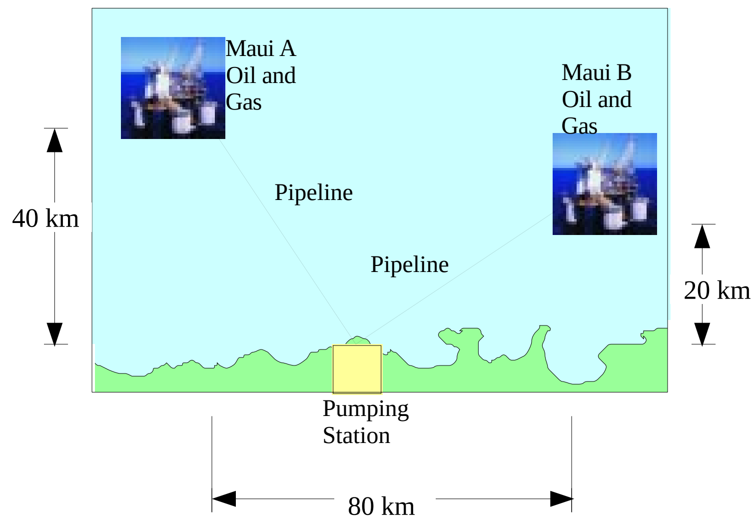
(a) What kind of function (linear exponential or power) does the best job of representing the relationship between the volume of the gas-filled portion of the cylinder (independent variable) and the pressure in the gas-filled portion of the cylinder (dependent variable)?

(b) Find an equation for the pressure as a function of the length of the gas-filled portion of the cylinder.

(c) The walls of the cylinder can withstand a pressure of up to 30 atmospheres. How far from the non-plunger end of the tube should the rod be placed?

(d) During part of the experiment, the students have to push the plunger into the cylinder at a steady rate of 0.5 centimeters per second. How rapidly is the pressure changing at the instant of time when the length of the gas-filled cylinder is 15 cm?

8 . The diagram given below shows part of the province of Taranaki on the west coast of the North Island of New Zealand. Taranaki has been nicknamed the “Energy Province” in New Zealand because reserves of petroleum and natural gas have been located of the Taranaki coast. The two main natural gas fields are called “Maui A” and “Maui B.” Spurred by the “oil shocks” of the late 1970’s the New Zealand government began to set up the pipelines and stations needed to extract natural gas and oil from the Maui fields. The most popular plan was to build two pipelines that ran to a central pumping station on shore.



The Maui A platform was located 40 km off the coast and the Maui B platform was 20 km off the coast. The distance between the two platforms (along the coastline) was 80 km. Where was the most cost-effective place to build the pumping station?

9 . Sleep deprivation is one of the most well-studied impediments to clear reasoning, sound judgment and academic performance. The subject has been widely studied by several branches of the military as soldiers (especially officers) often experience sleep-deprivation during stressful episodes such as combat operations. The military has been particularly interested in studying how sleep deprivation affects soldiers' ability to perform fairly complex information-processing tasks not unlike the problems that college students encounter on final examinations. The results from a very informal study conducted with a group of college students are shown in the table below.

Hours of Sleep	8	4	7	2	8	5
Score on Exam	96	24	74	6	92	40

(a) Find the equation of the linear function that best represents this data.

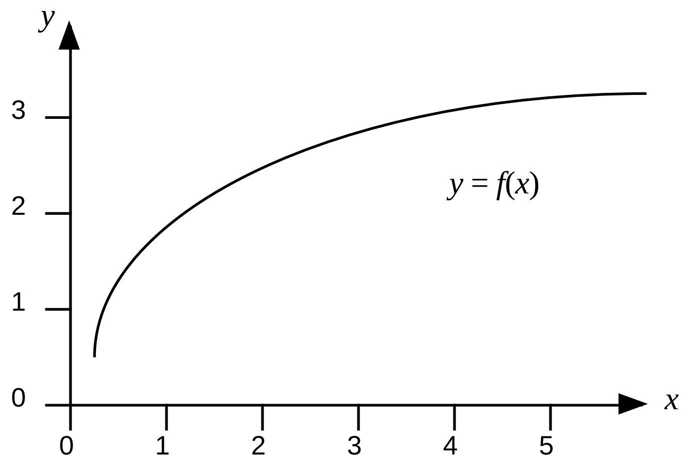
(b) Find the equation of the exponential function that best represents this data.

(c) Find the equation of the power function that best represents this data.

(d) Which of the three functions best represents the data ? Explain your answer.

(e) A student needs to get an 84 on the exam to pass the course. How many hours sleep should that student try to get?

10. Consider the function $y = f(x)$ graphed below. For each of the following pairs of numbers, decide which is larger.



(a) $f(3)$ or $f(4)$?

(b) $f(3) - f(2)$ or $f(2) - f(1)$?

(c) $\dfrac{f(2) - f(1)}{2 - 1}$ or $\dfrac{f(3) - f(1)}{3 - 1}$?

(d) $f'(1)$ or $f'(4)$?

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The domain of $g(x)$ is the interval [-4, 4].
1.(b) The x-coordinates of the points where the derivative of $f(x)$ is equal to zero are: x = -3, -2, -1, 0, 1, 2, 3. The derivative of $f(x)$ is not defined at either of the points $x = -4$ or $x = 4$ and so cannot be equal to zero at these points.
1.(c) $g'(x) = 2 \cdot f(x) \cdot f'(x) - 2 \cdot f'(x) = 2 \cdot f'(x) \cdot [f(x) - 1]$.
1.(d) Based on the answer to part (c), $g'(x) = 0$ when either $f'(x) = 0$ or when $f(x) = 1$. The points at which $g'(x) = 0$ are: x = -3.5, -3, -2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5, 3 and 3.5.
1.(e) Classifying the critical points from Part (d) is not all that easy, but you can do it with just the information contained in the graph of $y = f(x)$ and the equation for $g'(x)$. The method for classifying each critical points (as a maximum, a minimum or neither) that is used here is to look at the sign of $g'(x)$ just to the left and just to the right of the critical point. For example, for the critical point located at $x = -3.5$:

> **Just to the left of x = -3.5:** $f(x)$ is slightly larger than 1, so $[f(x) - 1]$ is positive. However, the graph of $y = f(x)$ is decreasing so $f'(x) < 0$. Therefore $g'(x)$ is equal to a positive times a negative and $g'(x)$ is negative just to the left of $x = -3.5$.

Just to the right of x = -3.5: $f(x)$ is slightly smaller than 1, so $[f(x) - 1]$ is negative. The graph of $y = f(x)$ is still decreasing so $f'(x) < 0$. Therefore $g'(x)$ is equal to a negative times a negative, and so $g'(x)$ is positive just to the right of -3.5.

Therefore, $g(x)$ has a local minimum at $x = -3.5$. The classification of each critical point is given in the table below.

Point	Classification	Point	Classification
x = –3.5	Local minimum	x = 0.5	Local minimum
x = –3	Local maximum	x = 1	Local maximum
x = –2.5	Local minimum	x = 1.5	Local minimum
x = –2	Local maximum	x = 2	Local maximum
x = –1.5	Local minimum	x = 2.5	Local minimum
x = –1	Local maximum	x = 3	Local maximum
x = –0.5	Local minimum	x = 3.5	Local minimum
x = 0	Local maximum		

2.(a) No the number of rabbits cannot be a linear function as the slope is not constant. Between 1859 and 1866 the slope is 2032.71 rabbits per year whereas between 1866 and 1869 the slope is 672,915.67 rabbits per year.

2.(b) No, the number of rabbits is not an exponential function as you get different values for the growth factor depending on what data points you use to calculate it. For example, if you use the points (0, 24) and (7, 14,253) you get a growth factor of $B = 2.490245414$. If you use the points (7, 14,253) and (10, 2,033,000) then you get a growth factor of $B = 5.224888551$.

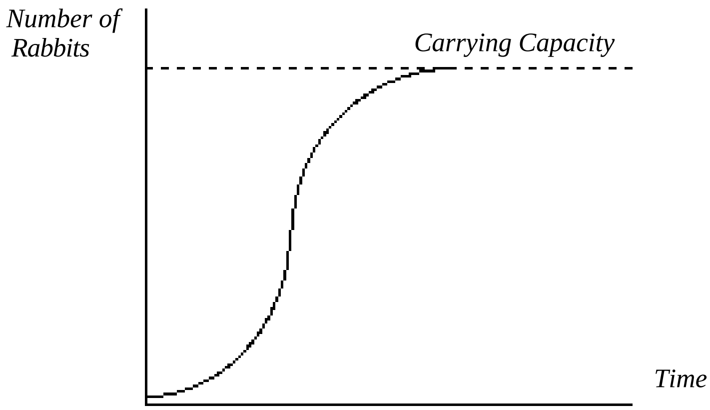
2.(c) No, the number of rabbits is not a power function as the graph of rabbits versus time does not pass through the origin, (0, 0).

2.(d) Of the three possibilities, exponential regression appears to give the correlation coefficient that is closest to 1. If N = number of rabbits and T = number of years since rabbits introduced then the exponential equation is:

$$N = 17.86013243033 \cdot (2.990772231941)^T.$$

2.(e) Substituting $T = 143$ into this equation gives approximately $1.88*10^{69}$ rabbits.

2.(f) Evaluating the exponential function for $T = 143$ gives about $1.9*10^{69}$ rabbits. At a mass of only 1 kg per rabbit, this means that there is 10^{29} times more mass (in the form of rabbits) roaming Mr. Austin's property than there is mass in the sun! The estimate given by the exponential function is therefore much too high. A more realistic scenario for the growth of the rabbit population could be that to begin with their numbers increased in a way that was approximated by exponential growth, but eventually leveled off due to limited space and resources at the "carrying capacity" of Mr. Austin's property. A graph of population versus time that shows this kind of growth is given below.



3.(a) $y = k\cdot(x + 3)(x - 1)(x - 4)$ where k is a negative number.
3.(b) $y = k\cdot(x + 3)(x - 1)(x - 4)$ where k is a positive number.
3.(c) $y = k\cdot(x + 2)(x - 1)(x - 3)(x - 4)$ where k is a positive number.
3.(d) $y = k\cdot(x + 2)(x - 1)(x - 3)^2$ where k is a negative number.

4.(a) The difference quotient is: $\dfrac{g(1+h)-g(1)}{h} = \dfrac{\frac{f(1+h)}{1+h} - \frac{f(1)}{1}}{h}$.

4.(b) Simplifying this difference quotient: $\dfrac{\frac{f(1+h)}{1+h} - \frac{f(1)}{1}}{h} = \dfrac{f(1+h)-(1+h)\cdot f(1)}{h\cdot(1+h)}$

Simplifying further gives:

$$\frac{f(1+h)-(1+h)\cdot f(1)}{h\cdot(1+h)} = \frac{1}{1+h}\cdot\frac{f(1+h)-f(1)}{h} - \frac{1}{1+h}\frac{h\cdot f(1)}{h}$$

4.(c) Taking the limit of this as $h \to 0$ gives: $g'(1) = f'(1) - f(1) = 1 - 1 = 0$.

4.(d) Using the quotient rule: $g'(1) = \dfrac{f'(1)\cdot 1 - 1\cdot f(1)}{1^1} = 0$.

5.(a) Defining $f(x)$ is pieces: $f(x) = \begin{cases} -1 & ,x < 1 \\ x-2 & ,x \geq 1 \end{cases}$.

5.(b) The derivative of $g(x)$ is given by the Product Rule: $g'(x) = f'(x)\cdot e^x + f(x)\cdot e^x$. So, in order for the derivative of $g(x)$ to be defined, both $f(x)$ and the derivative $f'(x)$ must both be defined. $f(x)$ is always defined, but the derivative $f'(x)$ is not defined at $x = 1$, as there is a "sharp corner" there. Therefore, the domain of the derivative $g'(x)$ consists of all real values of x except $x = 1$.

5.(c) Defining the derivative $g'(x)$ in pieces: $g'(x) = \begin{cases} -e^x & ,x < 1 \\ e^x + (x-2)\cdot e^x & ,x > 1 \end{cases}$

5.(d) Defining the derivative $g''(x)$ in pieces: $g''(x) = \begin{cases} -e^x & ,x < 1 \\ 2e^x + (x-2)\cdot e^x & ,x > 1 \end{cases}$

5.(e) The original function $g(x)$ will be concave up when the second derivative is positive and concave down when the second derivative is negative. Based on the equation for the second derivative found in Part (d), the original function $g(x)$ will be concave down when $x < 1$ and concave up when $x > 1$.

6.(a) No the curve cannot be the graph of a function. This is because the curve fails the vertical line test.

6.(b) $\frac{dy}{dx} = \frac{-(x-1)}{y}$.

6.(c) $\frac{dy}{dx} = \frac{-(x-1)}{\sqrt{1-(x-1)^2}}$.

6.(d) The two equations aren't really the same. The equation obtained in Part (b) works for any point (x, y) that lies on the curve (except (0, 0) and (2,0)) regardless of whether the point lies above the x-axis or below the x-axis. The equation for the derivative obtained in Part (c) only works when the point lies above the y-axis.

7.(a) Based on correlation coefficients, a power function does the best job.

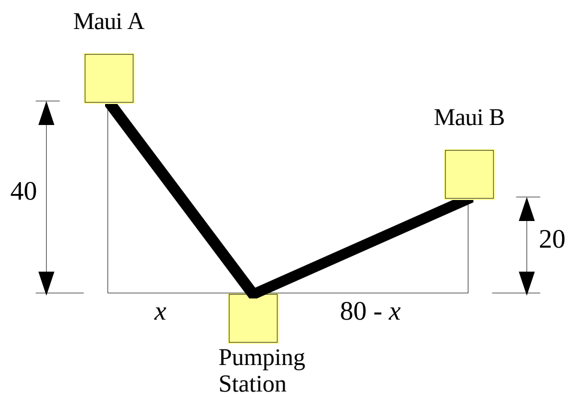
7.(b) Let V = volume of gas (in cubic cm) and P = pressure of gas (in atmospheres). Then from power regression on a calculator:

$$P = 636.56 \cdot V^{-1.42076}.$$

7.(c) The rod should be positioned about 2.7332 cm from the non-plunger end of the cylinder.

7.(d) At that instant of time, the rate of change of the pressure was about +0.12645 atmospheres per second.

8. The most cost-effective solution will probably be the one that involves the shortest possible pipelines. Therefore, the quantity that you should try to minimize here is the total length of the pipeline. If the variable x represents the distance (along the coastline) between the Maui A platform and the pumping station, then the situation can be represented as:



Using the Pythagorean Theorem the total length of the pipeline required to connect both platforms to the pumping station can be expressed as a function of x:

$$L(x) = \sqrt{40^2 + x^2} + \sqrt{20^2 + (80-x)^2}.$$

Differentiating with respect to x:

$$L'(x) = \frac{x}{\sqrt{40^2 + x^2}} - \frac{(80 - x)}{\sqrt{20^2 + (80 - x)^2}}.$$

Setting the derivative equal to zero and solving for x gives: x = 160/3 km. Therefore, in order minimize the total length of the two pipelines, the pumping station should be built 160/3 km from the Maui A platform (distance measured along the Taranaki coast).

9.(a) $s(t)$ = 15.295*t - 31.341 (correlation coefficient = 0.99032288)
9.(b) $s(t)$ = 3.449*e^(0.43*t) (correlation coefficient = 0.97573011)
9.(c) $s(t)$ = 1.539*t^1.9872 (correlation coefficient = 0.99950402)
9.(d) The power function - it has the correlation coefficient that is closest to 1.
9.(e) About 7.48 hours.

10.(a) $f(4)$.
10.(b) $f(2) - f(1)$
10.(c) $(f(2) - f(1))/(2 - 1)$
10.(d) $f'(1)$