Exam 2: Solutions

Problem 1:

The graph given below is the graph of the **derivative** of a function $f(x)$.

$y = f'(x)$

GRAPH OF
DERIVATIVE
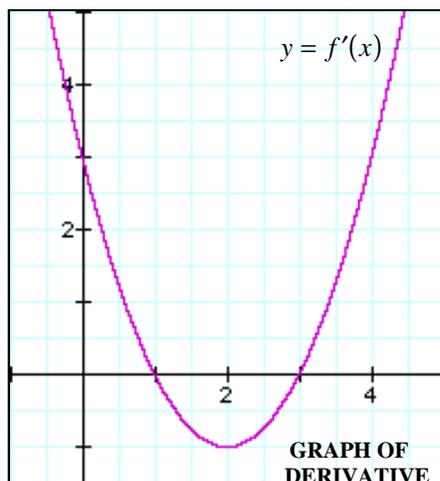
(a)   The critical points of the original function, $f(x)$, are the points where the first derivative, $f'(x)$ is equal to zero. The critical points of $f(x)$ are located at the same x-coordinates as the x-intercepts of the derivative, $f'(x)$. If you inspect the graph, then you can see that these occur at $x = 1$ and $x = 3$.

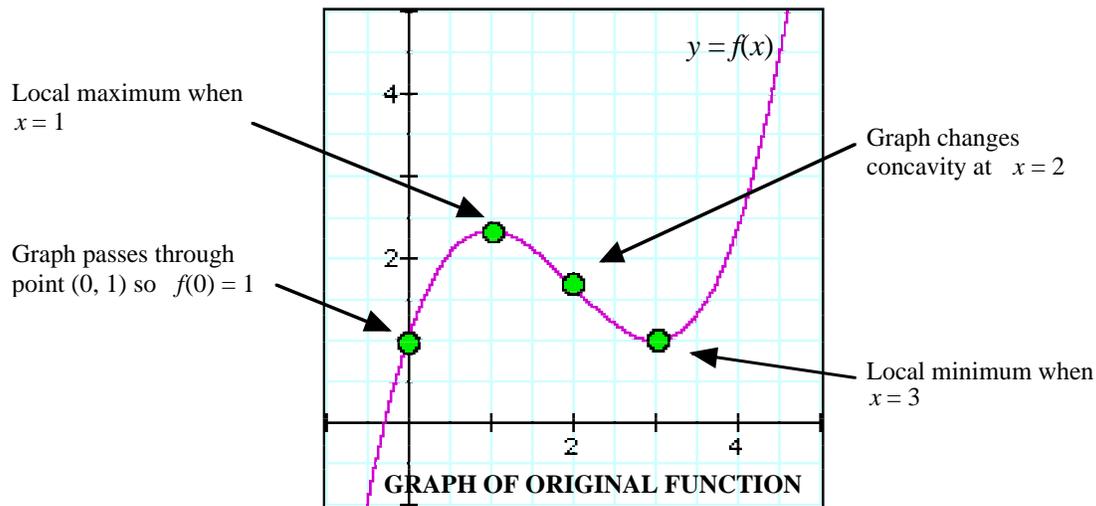**(b)**   **Classifying the critical points of $f(x)$:**

If you look at the graph of the derivative, you can see that slightly to the left of $x = 1$, the derivative is positive and slightly to the right of $x = 1$, the derivative is negative. When the first derivative goes from positive to negative, the critical point is a local maximum.

Similarly, from the graph of the derivative, if you look slightly to the left of $x = 3$ then you can see that the derivative is negative whereas slightly to the right of $x = 3$ the derivative is positive. When the first derivative goes from negative to positive the critical point is a local minimum.

(c)   The definition of a point of inflection is that this is where the graph of $f(x)$ changes concavity. You can locate this point from the derivative graph by following the following reasoning:

When the derivative is decreasing, the graph of the original function $f(x)$ is concave down. When the derivative is increasing, the graph of the original function $f(x)$ is concave up. The point(s) of inflection of $f(x)$ will occur at any point where the derivative graph goes from increasing to decreasing or vice versa. If you look at the derivative graph, you will be able to see that the only point where this occurs is at $x = 2$.

(d)   A possible[1] graph of the original function $y = f(x)$ is shown below.

Local maximum when $x = 1$

Graph changes concavity at $x = 2$

Graph passes through point $(0, 1)$ so $f(0) = 1$

Local minimum when $x = 3$

GRAPH OF ORIGINAL FUNCTION

Problem 2:

In this problem, the function $f(x)$ will always refer to the function defined by:

$$f(x) = \frac{-1}{x^2}.$$

(a)   One way to find the derivative $f'(x)$ is the re-write the function as: $f(x) = -1 \cdot x^{-2}$.

Differentiating this (with the power and constant multiplier rules) gives:

$$f'(x) = (-1)(-2) \cdot x^{-2-1} = 2 \cdot x^{-3} = \frac{2}{x^3}.$$

(b)   The tangent line to the graph of $y = f(x)$ at the point $(1, -1)$ is the line that goes through the point $(1, -1)$ with a slope equal to $f'(1)$.

You are looking for an equation of the form: $y = m \cdot x + b$.

Determining the Slope, m:

The slope of the tangent line is the value of the derivative at the point where the tangent line touches the curve $y = f(x)$. In this case:

$$m = f'(1) = \frac{2}{(1)^3} = 2.$$

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[1] There are many possible answers here, although correct answers will quite closely resemble the graph given. What we were looking for was that you included the point $(0, 1)$ on the graph, had the critical points and inflection point in the right places, indicated the locations of these special points and had a plausible shape for the graph overall.

Determining the Intercept, b:

To find the intercept, substitute the x- and y-coordinates of the point you are given and the slope $m = 2$ that you have just found into the equation for a line, $y = m \cdot x + b$.

In this case:
$$-1 = (2)(1) + b$$
so that $b = -3$.

The equation of the tangent line is then: $y = 2 \cdot x - 3$.

(c)   The slope of the line joining the points $(1, f(1))$ and $(1 + h, f(1 + h))$ is the rise over the run:
$$Slope = \frac{f(1+h) - f(1)}{1 + h - 1} = \frac{\frac{-1}{(1+h)^2} - (-1)}{h} = \frac{\frac{-1}{(1+h)^2} + 1}{h}.$$

Please note that I have simplified the expression slightly to aid readability - you did not have to simplify on the test.

(d)   The derivative $f'(1)$ is the limit of $\dfrac{f(1+h) - f(1)}{h}$ as $h \to 0$. The first step in finding this limit is to simplify the expression for the difference quotient as much as possible.

$$\frac{f(1+h) - f(1)}{h} = \frac{\frac{-1}{(1+h)^2} + 1}{h} = \frac{-1 + (1+h)^2}{h} = \frac{-1 + 1 + 2h + h^2}{h} = \frac{2h + h^2}{h} = 2 + h.$$

With the difference quotient simplified, you can now take the limit as $h \to 0$. The limit of $2 + h$ (as $h \to 0$) is equal to 2. Therefore, $f'(1) = 2$.

Problem 3:

The population of Hungary is represented by the equation:

$$P(t) = 10.8 \cdot (0.998)^T,$$

where P will always refer to the population of Hungary in units of millions of people and T will always refer to the number of years since 1990.

(a)   Since $P(T)$ is an exponential function, you use the rule for differentiating exponential functions:

$$\text{If: } f(x) = A \cdot B^x \text{ then: } f'(x) = A \cdot B^x \cdot \ln(B).$$

This gives that:

$$P'(T) = 10.8 \cdot (0.998)^T \cdot \ln(0.998) \approx -0.0216 \cdot (0.998)^T.$$

(b) Evaluating the equation from Part (a) for $T = 11$ gives:

$$P'(11) = -0.0216 \cdot (0.998)^{11} \approx -0.02115.$$

A practical interpretation of this number is that when T increases from $T = 11$ to $T = 12$, then the population of Hungary will decrease by approximately 0.02115 million people (i.e. about 21,150 people).

(c) Your objective in this part is to find the value of T that gives $P(T) = 5.8$. To do this you could set up the equation:

$$5.8 = 10.8 \cdot (0.998)^T.$$

Simplifying gives:

$$(0.998)^T = \frac{5.8}{10.8}.$$

Taking logs of both sides and re-arranging to make T the subject of the equation gives:

$$T \cdot \log(0.998) = \log\left(\frac{5.8}{10.8}\right)$$

$$T = \frac{\log\left(\frac{5.8}{10.8}\right)}{\log(0.998)} \approx 310.53.$$

So, in the year $(1990 + 310) = 2300$, the population of Hungary will reach 5.8 million people.

Problem 4:

In this problem, T will always represent the number of seconds since the Rob Neel fired his rifle. In this problem, $d(T)$ will always represent the height (in feet) of the bullets above the ground. The function d is defined by the equation given below.

$$d(T) = -16 \cdot T^2 + 2367 \cdot T + 5.$$

(a) As you learned in the lab with the motion detector, *velocity* is the derivative of the distance function. Therefore, you can obtain the velocity of the bullets by finding the derivative of $d(T)$:

$$Velocity = d'(T) = -32 \cdot T + 2367.$$

(b) When the bullets will reach maximum height, their velocity will equal zero. You can calculate when this occurs by finding the value of T for which $d'(T) = 0$. Solving this equation for T:

$$T = \frac{2367}{32} \approx 73.97 \ \text{ seconds after the rifle was fired.}$$

The maximum height that the bullets achieved can be found by substituting $T = 73.97$ into the equation for the function $d(T)$ (NOT the derivative). This gives:

$$\text{Maximum height} = d(73.97) \approx 87547.02 \ \text{ feet.}$$

(c) The process of completing the square for $d(T)$ is shown below. When completing the square you are ultimately trying to convert a quadratic equation into *vertex form* which resembles:

$$y = c \cdot (x - h)^2 + k,$$

where c, h and k are constants.

$$
\begin{aligned}
d(T) \ &= \ -16 \cdot T^2 + 2367 \cdot T + 5 \\[4pt]
&= \ -16 \cdot \left[T^2 - \tfrac{2367}{16} \cdot T - \tfrac{5}{16} \right] \\[4pt]
&= \ -16 \cdot \left[T^2 - \tfrac{2367}{16} \cdot T + \left(\tfrac{2367}{32}\right)^2 - \left(\tfrac{2367}{32}\right)^2 - \tfrac{5}{16} \right] \\[4pt]
&= \ -16 \cdot \left[\left(T - \tfrac{2367}{32}\right)^2 - \left(\tfrac{2367}{32}\right)^2 - \tfrac{5}{16} \right] \\[4pt]
&= \ -16 \cdot \left(T - \tfrac{2367}{32}\right)^2 + 16 \cdot \left(\tfrac{2367}{32}\right)^2 + 5 \\[4pt]
&= \ -16 \cdot \left(T - \tfrac{2367}{32}\right)^2 + 87547.02 \, .
\end{aligned}
$$

Problem 5:

In this problem, the function f will always refer to the function defined by the equation:

$$f(x) = \frac{1 - x}{1 - x^2}$$

and the function g will always refer to the function defined by the graph:

The function continues to oscillate in this way.
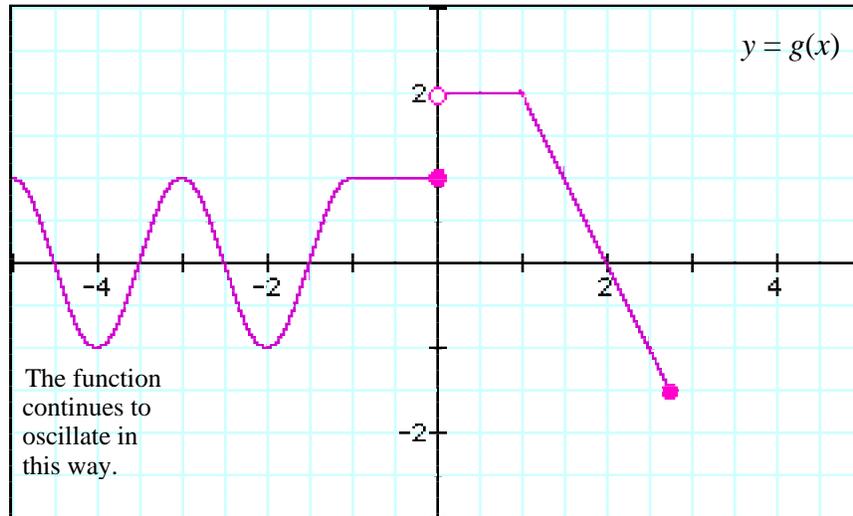
(a) $\dfrac{Lim\ f(x)}{x \to 1} = \dfrac{1}{2}.$

To deduce this limit, note that the denominator of f can be factored:

$$f(x) = \frac{1-x}{1-x^2} = \frac{1-x}{(1-x)\cdot(1+x)} = \frac{1-x}{1-x}\cdot\frac{1}{1+x}.$$

When x is close to 1, $f(x)$ will be very close to $(1)\cdot(1/2) = 1/2$.

(b) $\dfrac{Lim\ g(x)}{x \to 0}$ does not exist because the left hand limit (which is equal to 1) is not equal to the right hand limit (which is equal to 2).

(c) $\dfrac{Lim\ g(x)}{x \to -\infty}$ does not exist because the function g oscillates up and down, never settling down to approach a steady height.

(d) $\dfrac{Lim}{x \to 2^+}\dfrac{f(x)}{g(x)} = -\infty.$

To deduce this, you can look at the behavior of the functions f and g as each approach $x = 2$ from the right, and then combine these to understand how the quotient $\frac{f(x)}{g(x)}$ behaves as x approaches 2 from the right.

Behavior of $f(x)$ as x approaches 2 from the right.

The function f is defined when $x = 2$. Its value (which is equal to the limit from

the right) is: $f(2) = 1/3$. Therefore, when x is a number slightly larger than 2, $f(x)$ will be approximately equal to 1/3.

Behavior of $g(x)$ as x approaches 2 from the right.

Inspection of the graph of g shows that when x is a number slightly larger than 2, $g(x)$ is a negative number that is very, very close to zero.

Behavior of $\frac{f(x)}{g(x)}$ as x approaches 2 from the right.

Combining the previous two observations, you have that when x is a number slightly greater than 2,

$$\frac{f(x)}{g(x)} = \frac{\approx \frac{1}{3}}{-(small)} = -(big).$$

So, as x gets closer and closer to 2 from the right, $\frac{f(x)}{g(x)}$ is a larger and larger negative number.

Problem 6:

(a) The completed table is shown below.

| Length of Ride (Miles)	0.66	1.2	NOT POSSIBLE	7.2
Cost ($)	**2.25**	**3.25**	13.00	**15.25**

(b) There are many possible, acceptable ways to write down the equation for the function that gives the cost of the taxi ride (and is valid between 0 miles and 1 mile). One option is to write the equation in pieces. This is shown below.

Let C represent the cost of the taxi ride (in dollars) and x represent the length of the ride (in miles).

$$C(x) = \begin{cases} 1.25 & , 0 < x \leq 0.25 \\ 1.75 & , 0.25 < x \leq 0.5 \\ 2.25 & , 0.5 < x \leq 0.75 \\ 2.75 & , 0.75 < x \leq 1 \end{cases}$$

Problem 7:

The diagram given below shows a "bird's eye" view of Jethro's corral, with the dimensions of length (L) and width (w) indicated.
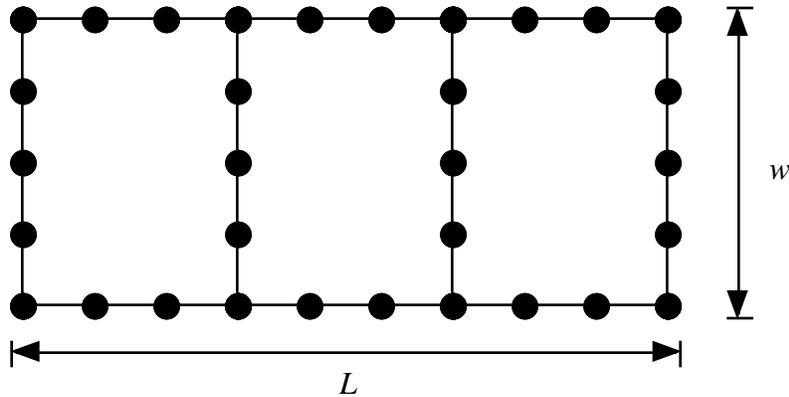
(**a**)   The area is the length times with width of the corral: $A = L \cdot w$.

The plan for the corral calls for two long sides (each of length L) and four shorter sides (each of length w). The total number of feet of fencing material required for the corral is therefore:

$$2 \cdot L + 4 \cdot w.$$

(**b**)   Jethro has 600 feet of fencing material available to him. Therefore:

$$2 \cdot L + 4 \cdot w = 600.$$

Rearranging this equation to make L the subject gives:

$$L = 300 - 2 \cdot w.$$

Using this equation to replace the L in the equation for the area enclosed gives:

$$A = L \cdot w = w \cdot (300 - 2 \cdot w) = 300 \cdot w - 2 \cdot w^2.$$

(**c**)   To calculate the dimensions (i.e. length and width) for the corral that will enclose the most area, you can find the derivative of A with respect to w, set the derivative equal to zero and then solve for w. Doing this gives:

$$A' = 300 - 4 \cdot w = 0$$

so that $w = 75$. Using the derivative to check that this point really does correspond to a *maximum*:

w	74.9	75.1
Derivative	+0.4	−0.4

As the derivative is positive to the left and negative to the right of $w = 75$, it is safe to conclude that $w = 75$ gives maximum area. Note that when $w = 75$, the corresponding length, L, is $L = 150$.