Homework Assignment 14: Solutions

1. The initial fish population can be calculated by substituting $t = 0$ into the equation,

$$N(t) = a + 200 \cdot e^{-t}.$$

This gives an initial fish population of $a + 200$ fishes. As t increases, the number of fish in the tank will **decrease**. You can see that this is what will happen by examining the behavior of the equation for $N(t)$ term by term.

- First term, a: This is a constant and will be equal to a regardless of what the value of t is.
- Second term, $200 \cdot e^{-t}$: Re-writing this without the negative sign in the exponent gives:

$$200 \cdot e^{-t} = 200 \cdot \left(\frac{1}{e}\right)^t.$$

The numerical value of the special number e is about 2.72. The really important fact about the special number e is that it is greater than one, as this means that the second term of $N(t)$ will be a decreasing exponential function.

Therefore, overall,

$$N(t) = \text{(constant)} + \text{(decreasing exponential function)}$$

so in the short term, I would expect the numbers of fish in the tank to decrease from their initial level of $200 + a$ fishes.

2. In the long term, the first term of $N(t)$ will stay constant, and the second term of $N(t)$ will get closer and closer to zero. Therefore, in the long term, $N(t)$ will resemble:

$$N(t) = \text{(constant)} + \text{(quantity getting closer and closer to zero)}$$

so in the long term, I would expect the fish population to decrease until there were approximately a fish left in the tank. The population should stabilize at this level.

3. There are two reasonably sensible predictions that you could make about the appearance of the graph of $y = f(x)$ near $x = 2$. Recall that f is the function defined by the equation:

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}.$$

Either of these predictions is acceptable, so long as it is accompanied by:

(I) An analysis of the algebraic structure of the equation for $f(x)$, and,

(II) A graph that is consistent with that analysis.

- **Prediction 1:** Notice that the denominator of the equation for $f(x)$,

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}$$

includes the factor $(x - 2)$. This means that when x is near 2, evaluating $f(x)$ will require division by a very small number that is close to zero. When you divide a quantity by a number that is very close to zero, it makes the result very large overall. Based on this analysis, I would predict that the graph of $y = f(x)$ should show a vertical asymptote at $x = 2$.

Based on this prediction, the graph of $y = f(x)$ could resemble any of the following.

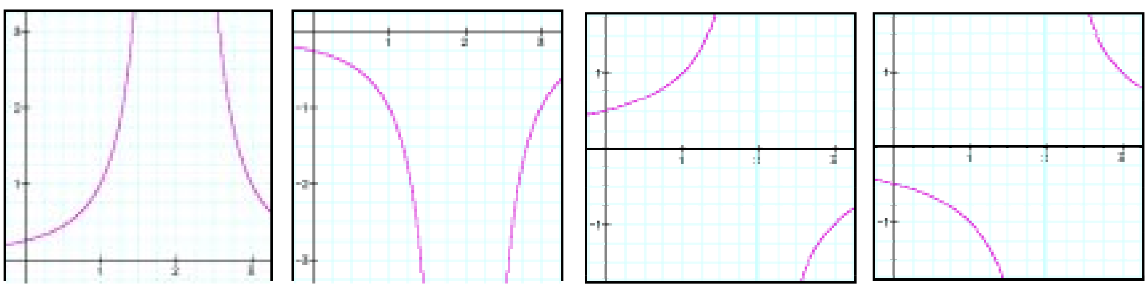

Figure 1(a, b, c, d): Possible appearances of the graph of $y = f(x)$ near $x = 2$, based on the prediction that $y = f(x)$ will have a vertical asymptote at $x = 2$.

- **Prediction 2:** Notice that the denominator of the equation for $f(x)$,

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}$$

includes the factor $(x - 2)$. This means that when x is near 2, evaluating $f(x)$ will require division by a very small number that is close to zero. However, also note that when you substitute $x = 2$ into the numerator of the equation for $f(x)$, you get zero. Therefore, when x is near 2, both the numerator and the denominator of the equation for $f(x)$ are both close to zero. Ordinarily, when you divide by something that is very close to zero, it makes the quotient very large overall. However, when the numerator of the quotient is also a tiny number, the result of dividing by something close to zero may not be too large. Based on this reasoning, I would expect that the graph of $y = f(x)$ would have a hole in it near $x = 2$, but that there would be no vertical asymptote.

Based on this prediction, the graph of $y = f(x)$ could resemble the following.

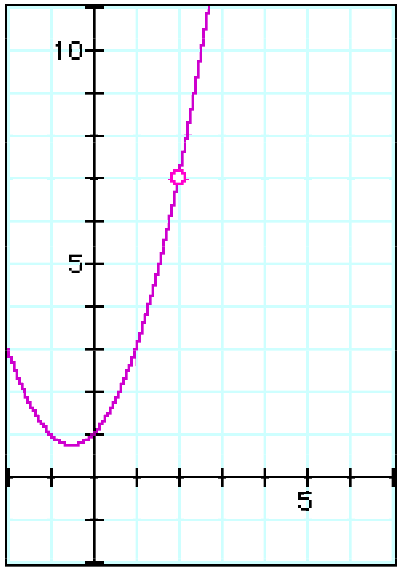

Figure 2: Possible appearance of $y = f(x)$ based on prediction of no vertical asymptote near $x = 2$.

4. The graph from a TI-83 is shown below. This graph clearly shows that f is not defined at $x = 2$ as there is a hole in the graph there. There is no vertical asymptote at $x = 2$. This situation appears to correspond to the second prediction made above - that the effect of the small numerator will compensate for the small denominator, and the graph of $y = f(x)$ will not have a vertical asymptote at $x = 2$.

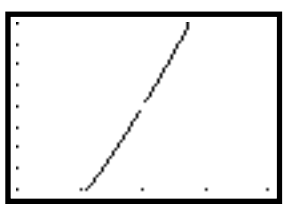

Figure 3: Graph of $y = f(x)$ from TI-83.

5. Using the fact that:

$$(x-2)\cdot\left(x^2+x+1\right)=x^3-x^2-x-2$$

you can re-write the equation for *f*(*x*) as follows:

$$f(x)=\frac{x^3-x^2-x-2}{x-2}=\frac{(x-2)\cdot\left(x^2+x+1\right)}{(x-2)}=\frac{(x-2)}{(x-2)}\cdot\left(x^2+x+1\right).$$

When *x* is near 2, the value of *f*(*x*) will be approximately equal to:

$$f(x)=\frac{(x-2)}{(x-2)}\cdot\left(x^2+x+1\right)\approx 1\cdot\left(2^2+2+1\right)=7.$$

So, as *x* approaches 2 from either the left or the right, the value of *f*(*x*) will be close to 7. In terms of the appearance of the graph, this means that instead of having vertical asymptotes that shoot up to +∞ or plunge down to –∞, the graph of *y* = *f*(*x*) will simply get close to a height of 7 as *x* gets close to a value of 2. (See the values of *f*(*x*) obtained by tracing the graph below.)

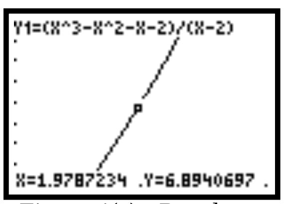

Figure 4(a): Result of tracing on the graph of *y* = *f*(*x*) slightly to the left of *x* = 2.

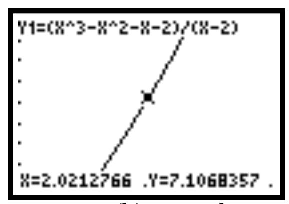

Figure 4(b): Result of tracing on the graph of *y* = *f*(*x*) slightly to the right of *x* = 2.