

### Homework Assignment 15: Solutions

1. To create a function that would approximate the population of Kenya as a function of time, I first entered the data into a graphing calculator and made a STATPLOT to see if there were any distinct patterns or trends in the data.

To make things slightly easier for the calculator, I entered “years since 1900” as the independent variable, rather than just entering the year. You can see the numbers that I entered and the plot that the calculator produced in Figures 1(a) and 1(b) below.

L1	L2	L3	2
0	1.84E6	-----	
10	2.5E6		
20	3.39E6		
30	4.61E6		
40	6.27E6		
50	8.12E6		
60			
L2(1)=1352000			

Figure 1(a): Some of the data entered into the calculator.

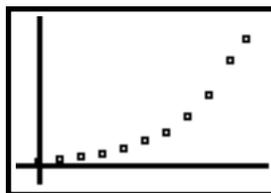


Figure 1(b): STAT-PLOT produced by a graphing calculator.

The overall pattern revealed in the plot is an increasing, concave up function. Often, exponential and power functions (where the power is greater than 1) do a reasonable job of representing these kind of relationships. Trying exponential and power regression on a calculator gave the results shown in Figure 2, below.

ExpReg
y=a*b^x
a=1316498.846
b=1.032077091
r^2=.9985952116
r=.999297359

Figure 2(a): Results of performing an exponential regression on the data.

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Figure 2(a): Results of performing a power regression on the data.

Even when the data is modified<sup>1</sup> so that the power regression does not encounter a domain error, the results of power regression are quite disappointing. (For example, the correlation coefficient is only about 0.84.) Based on these results, the exponential function:

<sup>1</sup> The main thing that is upsetting the calculator with the power regression is the fact that  $x = 0$  is one of the data points. A simple modification of the data is to add one to each of the  $x$ -values, so that now  $x$  represents the number of years since 1899, rather than the number of years since 1900. Since  $x = 0$  is no longer a data point, the calculator will happily perform power regression.

$$P = 1316498.846 * (1.032077091)^T$$

(where  $P$  is the population of Kenya and  $T$  is the number of years since 1900) seems to do the best job of representing the relationship between time and population.

2. The completed table is shown below. Using the function found in Question 1, the difference quotient will be:

$$\frac{1316498.846 * (1.032077091)^{60+h} - 1316498.846 * (1.032077091)^{60}}{h},$$

where  $h$  is a very small number. Note that “60” appears in the exponent as the equation from Question 1 used “years since 1900” as the independent variable, rather than simply just the year.

To complete the entries in the table, you evaluate the difference quotient given above for the values of  $h$ .

h	-0.1	-0.001	-0.000001	0.000001	0.001	0.1
Difference Quotient at t=1960	275920.65	276352.11	276357.4	276355.6	276360.83	276793.20

3. The slope of a tangent line is equal to the instantaneous rate of change of the function at the point where the tangent line is based (see Figure 3 below). The instantaneous rate of change of a function is equal to the limiting value of the difference quotient as  $h \rightarrow 0$ .

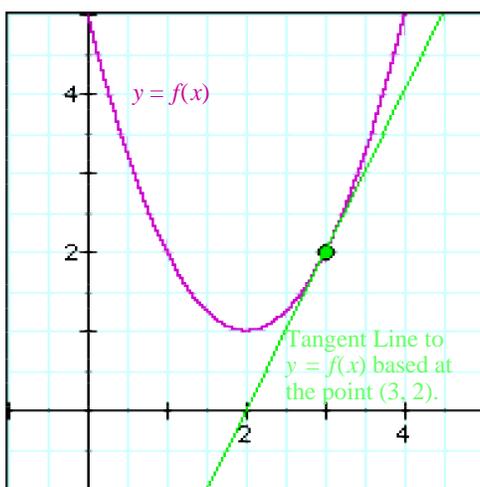


Figure 3: The graph of a function and the tangent line to the function based at the point (3, 2).

From the table in Question 2, the limiting value of the difference quotient (as  $h \rightarrow 0$ ) will be somewhere between 276357.4 and 276355.6. Averaging these gives a figure that will be close to the actual value of the limiting value (i.e. close to the actual slope of the tangent line):

$$\text{Slope} = \frac{276357.4 + 276355.6}{2} = 276356.5.$$

The equation for a tangent *line* will be a linear function, that is, a function of the form:

$$y = m \cdot T + b.$$

The slope of the tangent line,  $m$ , is:  $m = 276356.5$ . To find the intercept of the tangent line, note that the tangent line just brushes the graph of the population of Kenya at the point where  $T = 60$ . At this point, the height of the population graph is:

$$P = 1316498.846 \cdot (1.032077091)^{60} = 8752835.496.$$

Substituting  $T = 60$ ,  $m = 276356.5$ , and  $y = 8752835.496$  into the equation for the tangent line gives:

$$8752835.496 = (276356.5) \cdot (60) + b,$$

so that:  $b = -7828554.504$  and the equation for the tangent line is

$$y = 276356.5 \cdot T - 7828554.504.$$

A graph showing both the exponential function and the tangent line is given as Figure 4 (below).

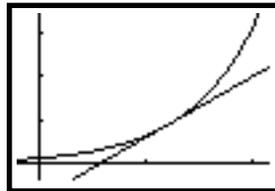


Figure 4: Exponential function representing population of Kenya and tangent line based at  $T = 60$ .

4. For 1920, the appropriate difference quotient will be:

$$\frac{1316498.846 \cdot (1.032077091)^{20+h} - 1316498.846 \cdot (1.032077091)^{20}}{h},$$

where  $h$  is a very small number. Note that “20” appears in the exponent as the equation from Question 1 used “years since 1900” as the independent variable, rather than simply just the year.

To complete the entries in the table, you evaluate the difference quotient given above for the values of  $h$ .

$h$	-0.1	-0.001	-0.000001	0.000001	0.001	0.1
Difference Quotient at $t=1920$	78036.66	78158.689	78160	78159.7	78161.15	78283.44

The estimate for the derivative or instantaneous rate of change of population at  $T = 20$  is then:

$$\text{Slope} = \frac{78160 + 78159.7}{2} = 78159.85.$$

Performing similar calculations for 1940 and 1980 gives the results summarized in the table given below.

Year	$T$	Instantaneous rate of change or derivative of population
1920	20	78159.85
1940	40	146969.4
1960	60	273656.5
1980	80	519652.5

5. Finally, we end as we began by trying to find an equation for a set of data points. Entering the data (see Figure 5(a)) into a calculator and graphing (see Figure 5(b)) shows a concave up, increasing relationship between  $T$  and the instantaneous rate of change.

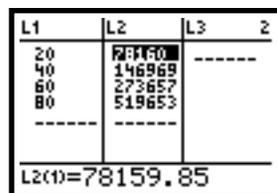


Figure 5(a): Data entered into calculator.

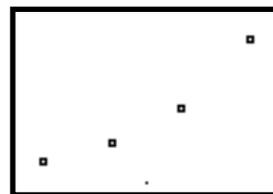


Figure 5(b): STAT-PLOT of instantaneous rate of change.

As noted previously, both exponential and power functions (when the power is greater than one) can do a reasonable job in these situations. Trying exponential and power regression gives the results shown in Figure 6 (below).

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ExpReg
y=a*b^x
a=41566.23296
b=1.032026464
r^2=.999966048
r=.9999830239
```

Figure 6(a): Results of performing exponential regression on data.

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PwrReg
y=a*x^b
a=1317.817364
b=1.32643821
r^2=.9597332424
r=.9796597585
```

Figure 6(b): Results of performing power regression on data.

Based on these results, the exponential function<sup>2</sup> seems to do a better job of representing the relationship because the correlation coefficient is closer to one. So, the equation<sup>3</sup> for the instantaneous rate of change as a function of time is:

$$\text{Instantaneous Rate of Change of Population} = 41566.233*(1.032026464)^T.$$

*with Respect to Time*

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<sup>2</sup> As you will learn in the next few weeks, this is no accident. One of the things that you will learn is that the derivative of an exponential function is another exponential function.

<sup>3</sup> If you look closely, you will see that the growth factor of this function (about 1.032) and the growth factor for the population function (also about 1.032) are remarkably similar. Again, this is no accident.