

### Homework Assignment 16: Solutions

Remember that in this homework assignment, the function  $f(x)$  will always refer to the function defined by the equation:

$$f(x) = \frac{1-x}{x^2},$$

with domain  $x > 0$ .

1. The completed table is given below.

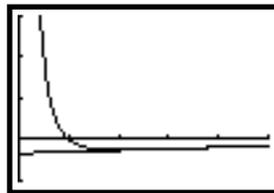
x	2.9	2.99	2.999	2.9999	3.0001	3.001	3.01	3.1
$\frac{f(x) - f(3)}{x - 3}$	0.03699	0.037037	0.037037	0.037037	0.037037	0.037037	0.037037	0.03699

Based on the trend of the values in the table, the limiting value of  $\frac{f(x) - f(3)}{x - 3}$  as  $x$  approaches 3 appears to be equal to. Therefore the value of the derivative of  $f(x)$  at  $x = 3$  is approximately 0.037037.

2. The slope of the tangent line is 0.037037, and the tangent line passes through the point  $(3, f(3)) = (3, -2/9)$ . The equation of the line with slope 0.037037 that passes through the point  $(3, -2/9)$  is:

$$y = 0.037037x - 0.3333332222.$$

The graph of  $y = f(x)$  and the graph of the tangent line is shown below.



3. There are four steps that one normally needs to complete in order to calculate the derivative algebraically:

- (a) Use the definition of the derivative to correctly find  $f(a + h)$ .
- (b) Use the definition of the function to calculate the difference quotient,  $\frac{f(a + h) - f(a)}{h}$ .

- (c) Simplify the difference quotient as much as possible. A good clue that you have done this is that there should be no factors of  $h$  remaining in the denominator of the difference quotient.
- (d) Find the limiting value of the difference quotient as  $h \rightarrow 0$ .

Next, I will carry out these steps, one after the other.

- (a) Formulating  $f(a + h)$ :

$$f(x) = \frac{1 - (a + h)}{(a + h)^2}.$$

- (b) Calculating the difference quotient:

$$\frac{f(a + h) - f(a)}{h} = \frac{\frac{1 - (a + h)}{(a + h)^2} - \frac{1 - a}{a^2}}{h}.$$

- (c) Simplifying the difference quotient gives:

$$\frac{f(a + h) - f(a)}{h} = \frac{\frac{1 - (a + h)}{(a + h)^2} - \frac{1 - a}{a^2}}{h} = \frac{\frac{a^2 - a^2 \cdot (a + h) - (a + h)^2 + a \cdot (a + h)}{a^2 \cdot (a + h)^2}}{h} = \frac{a^2 h - 2ah - h^2 + ah^2}{ha^2(a + h)^2}.$$

and simplifying further gives:

$$\frac{f(a + h) - f(a)}{h} = \frac{a^2 - 2a - h + ah}{a^2(a + h)^2}.$$

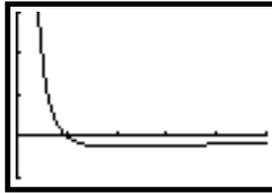
Note that the denominator of the difference quotient no longer contains any *factors* of  $h$ . There is an  $h$  still in the denominator, but it is not a factor by itself.

- (d) Taking the limit as  $h \rightarrow 0$  gives that the derivative of  $f(x)$  at  $x = a$  is equal to:

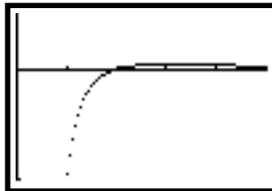
$$f'(a) = \frac{a^2 - 2a}{a^4} = \frac{1}{a^2} - \frac{2}{a^3}.$$

4. The two graphs ( $y = f(x)$  and  $y = f'(x)$ ) are shown on the next page.

Graph of  $y = f(x)$ :



Graph of  $y = f'(x)$ :



5. The correspondences between the behavior of the original function and the behavior of the derivative are summarized in the table shown below.

Behavior of original function/appearance of graph of $y = f(x)$	Behavior of derivative/appearance of graph of $y = f'(x)$
Increasing (as graph read from left to right)	Derivative is positive
Decreasing (as graph read from left to right)	Derivative is negative
Top of hill (maximum value) or bottom of valley (minimum value)	Derivative is zero
Concave up	Derivative is increasing
Concave down	Derivative is decreasing
Inflection point (concavity changes)	Derivative graph has a maximum value (top of a hill) or a minimum value (bottom of a valley)

For the specific example of  $f(x) = \frac{1-x}{x^2}$ :

Original Function	Derivative	Points or intervals
Increasing	Positive	$2 < x$
Decreasing	Negative	$0 < x < 2$
Max. or min.	Zero	$x = 2$
Concave up	Increasing	$0 < x < 3$
Concave down	Decreasing	$3 < x$
Inflection point	Max. or min.	$x = 3$