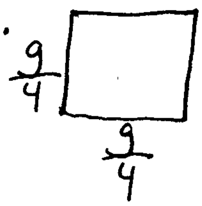
Sol'n Set 23

9. We'll call l the length & g the girth. We know from a previous problem that the area for a given perimeter is maximized in a square, so it would be safe to assume here that the cross-section is a square.



Thus the cross-sectional area $= \frac{g^2}{16}$.

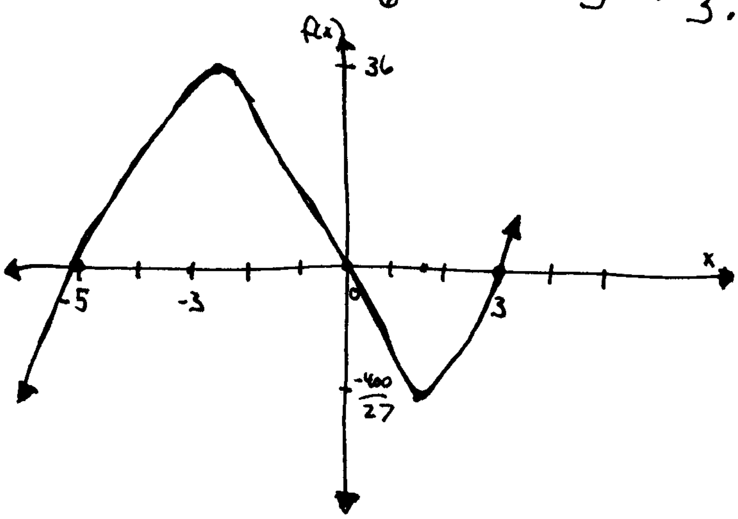
So $l + g \leq 108$, and the volume $V = l \cdot \frac{g^2}{16}$

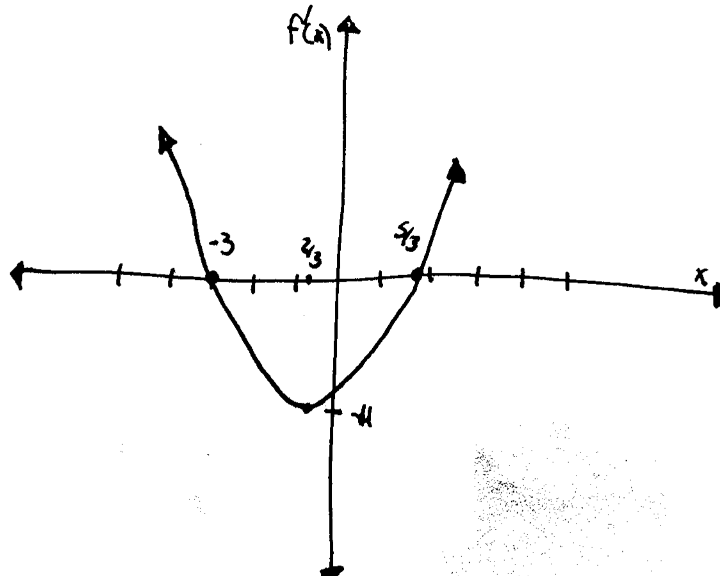
$l = 108 - g \Rightarrow V = (108-g)\frac{g^2}{16}$. This is maximized when $\frac{d}{dg}(V) = 0 \Rightarrow \frac{27}{2}g - \frac{3}{16}g^2 = 0 \Rightarrow g = 0$ or 72. $g = 0$ obviously isn't a maximum, so $g = 72$, $l = 36 \Rightarrow \boxed{V = 11{,}664 \text{ in}^2}$

11. a) $f(x) = x(x-3)(x+5)$ has zeros at $x = 0, 3, -5$

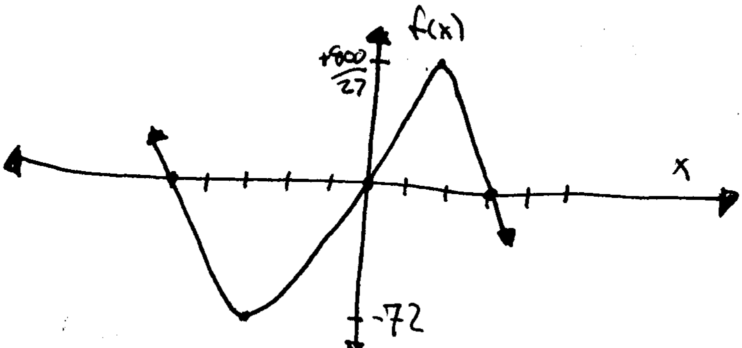
$f'(x)$ has zeros where $\frac{d}{dx}(x^3 + 2x^2 - 15x) = 0 \Rightarrow 3x^2 + 4x - 15 = 0$

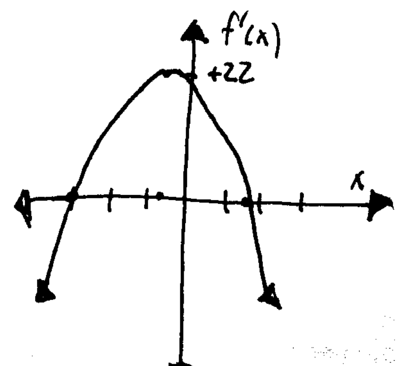
So $x = \frac{-4 \pm \sqrt{16 - 4 \cdot 3 \cdot (-15)}}{6} = -3$ or $\frac{5}{3}$.





b) This is part a), flipped upside down & stretched by 2





2\. a) $P'(x)$ has degree 5 $(x^2 \cdot (x^3 + \ldots + 8))$, so $P(x)$ has $\boxed{\text{degree } 6}$, since $\frac{d}{dx}(x^n) = n x^{n-1}$!

b) Critical points are where $P'(x) = 0$, which is when $\boxed{x = 0 \text{ or } -2}$

c) $P'(x)$ is negative for $x < -2$ & positive for $x > 2$, so this implies $P(x)$ is decreasing for $x < -2$ & increasing after, so it must hit $\boxed{\text{a minimum at } x = -2}$. We can't know what the minimum value is because $P(x)$ and $P(x) + C$ (where C is a constant) have the same derivative, and these have different minimal values.

3\. The maximum # of roots a polynomial can have is determined by the degree of the polynomial. If $p(x)$ has degree n, then $p''(x)$ has degree $n-2$, So it can have at most $n-2$ roots. Since points of inflection are where the 2nd derivative $= 0$, there can be at most $\boxed{n-2 \text{ points of inflection.}}$

5\. a) False, -3 makes it go to $-\infty$.

b) True, degree of 4 makes $-\infty$ positive, but -3 makes it negative again.

c) False, a poly can have complex roots.

d) True, since the derivative poly has limits at $-\infty$ and ∞, it must cross 0 & so $p(x)$ must have a turning point.

e) False, $p'(x)$ could go from $-$ to $+$ only once, as per $-12x^3$

f) False, critical points mean $p'(x) = 0$, and $p'(x)$ can have only 3 roots.

g) True, we know $P(x)$ goes to $-\infty$ on both ends, so it must hit a maximum in between.

h) False, since $\lim_{x \to -\infty} P(x) = -\infty$

10. Here's our info: $P(x)$ has zeros at -3 & $1 \Rightarrow$ factors of $(x+3)$ & $(x-1)$
$P'(x)$ has zeros at 1 & -1. $P(x)$ & $P'(x)$ both have roots at 1, so we have a double root of $(x-1)$. $P''(x)=0$ at $x=-3,-1.5,0$, so it must be of degree at least 3, so $P(x)$ must be at least of degree 5.
Since we have only 3 real roots, we need to throw in something with an imaginary root, like x^2+1.

So we guess $P(x)=k(x+3)(x-1)^2(x^2+1)$. $P(0)=1 \Rightarrow k$ must be $\frac{1}{3}$. So $\boxed{P(x)=\frac{1}{3}(x+3)(x-1)^2(x^2+1).}$