Exam 1: Solutions

1. The brief answers to this problem are:

    (A) Graph II   **(B)** Graph III
    (C) Graph I   **(D)** Graph IV.

More extensive answers are given below.

(A) The equation in Part A was: $y = f(2 \cdot x) + 1$. Working from the "inside of the formula out" gives that the two transformations that were performed were (i) a horizontal stretch that compressed the graph to half its original width, and (ii) a vertical shift up by 1. The graph that showed both of these features was Graph (II).

(B) The equation in Part B was: $y = f(\frac{1}{2} \cdot x) + 1$. Working from the "inside of the formula out" gives that the two transformations that were performed were (i) a horizontal stretch that expanded the graph to twice its original width, and (ii) a vertical shift up by 1. The graph that showed both of these features was Graph (III).

(C) The equation in Part C was: $y = f(x + 1) - 1$. Working from the "inside of the formula out" gives that the two transformations that were performed were (i) a horizontal translation that moved the graph one unit to the right, and (ii) a vertical shift down by 1. The graph that showed both of these features was Graph (I).

(D) The equation in Part D was: $y = f(x - 1) - 1$. Working from the "inside of the formula out" gives that the two transformations that were performed were (i) a horizontal translation that shifted the graph to the right by one unit, and (ii) a vertical shift down by 1. The graph that showed both of these features was Graph (IV).

2.(a) Based on the information given in the table, "debt" is not a perfectly linear function. The reason for this conclusion is the fact that you get different slopes (or average rates of change) if you use different pairs of points from the table. For example, using the points (1960, 100) and (1970, 115) gives the slope:

$$\text{Slope} = \frac{115 - 100}{1970 - 1960} = 1.5.$$

However, if you use the points (1970, 115) and (1980, 119) you get a different slope.

$$\text{Slope} = \frac{119 - 115}{1980 - 1970} = 0.4.$$

2.(b) Here, y will always represent the "debt" (as a percentage of income). There are several reasonable choices for the independent variable, x, however, and the particular choice that you make will affect the equation that you obtain when performing linear regression on your calculator. The two choices for the independent variable that are most like other problems that you have done in the course (and the resulting linear functions) are given below.

If the independent variable, x, is simply the year: $y = 1.48{\cdot}x - 2802.5$.

If the independent variable, x, is the number of years since 1960:

$$y = 1.48{\cdot}x + 98.3.$$

2.(c) Substituting the appropriate value of x into the linear equation gives a "debt" of 160.46%.

2.(d) The quantity used for "debt" can be represented symbolically by the following equation:

$$\text{"Debt"} = \frac{\text{Amount of money owed}}{\text{Income}} \times \frac{100}{1}.$$

If the amount of money owed is frozen, then the numerator in the function will remain constant. If you then get a cut in pay, then the denominator of this function will be reduced. Reducing the size of the denominator will have the effect of increasing the overall size of the fraction. Therefore, if your creditors froze the amount of money that you had borrowed, and you were forced to take a cut in pay, then your "debt" would rise.

3.(a) A plot of the data points (using the same sized axes as you were provided with) is shown below.

3.(b) Of the three options that you were given on the test,

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LinReg
y=ax+b
a=9.279757392
b=453.161486

■
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QuadReg
y=ax²+bx+c
a=-12.68123187
b=167.3288633
c=222.2073147
■
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```
ExpReg
y=a*b^x
a=402.291826
b=1.024525534

■
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only the quadratic function could plausibly represent the trend in the data. This is because only the quadratic can rise to a peak and then decrease afterwards – both the

linear and the exponential functions that are given continue to rise at all times, which is certainly not what the data plotted above shows. (Thank goodness, when you think about the quantity that is plotted above!).

3.(c) The effect of the use of the drug zidovudine on the number of cases of perinatal transmission of HIV from mother to child should be to reduce the number of such cases. Therefore, when the drug became widely used, we would expect the graph of number of new cases of perinatal transmission to stop increasing and start decreasing at about this time. From the values given in the table, this occurred somewhere between 1990 and 1993.

3.(d) Perinatal transmission of HIV from mother to child would be eliminated in the United States when the quadratic function from Question 3(b) reached zero – that is, when its graph crossed the x-axis. If you graph the quadratic function from part (b) on a calculator and then use the INTERSECT capability of the calculator to locate the point where the graph of the quadratic function crosses the x-axis you get about $x = 14.41$. This means, that according to the quadratic function, perinatal transmission of HIV from mother to child in the United States could have been eliminated as early as 1999.

4.(a) The only table that corresponded to the three pictures was the first table,

t	0	24	48
Number	1	2	4

4.(b) The table does show a perfectly exponential relationship. Irrespective of which pair of points are chosen to calculate the equation for the exponential function, you always obtain:

$$y = 1 \cdot (1.029302237)^x.$$

4.(c) To determine when the bacterial population reached 1000 bacteria, we need to solve the following equation for x:

$$1000 = (1.029302237)^x.$$

You can solve this equation for x either by graphing and finding the intersection point on a calculator, or (as x is in the exponent and nowhere else in the equation) use logarithms. Applying logarithms to both sides of the equation, and using the super-fun-happy-rule (SFHR) gives:

$$\log(1000) = x \cdot \log(1.029302237)$$

so that:

$$x = \frac{\log(1000)}{\log(1.029302237)} = 239.1788.$$

5. The information furnished to help with Euler's method was as follows:

- The independent variable, T, is the number of years since the WWE's initial public offering (IPO) of stock in 1999.
- The dependent variable, W, is the net worth of the WWE expressed in units of billions of dollars.
- The initial value of W is \$2.3 billion. In math symbols: $W(0) = 2.3$.
- The rate of change of W with respect to T is given by the equation: $\dfrac{\Delta W}{\Delta T} = 0.13 \cdot W(T) - 0.194$.
- The time interval, ΔT, is two years.

(a) The completed table generated by doing Euler's method with the information given above is shown below.

Time (T in years)	Net worth of WWFE ($ billions)	Rate of change (in billions of dollars per year)	Amount net worth of WWFE will change in the next two years	New net worth of WWFE ($ billions)
T=0	2.3	0.105	0.21	2.51
T=2	2.51	0.1323	0.2646	2.7746
T=4	2.7746	0.166698	0.333396	3.107996
T=6	3.107996	0.21003948	0.42007896	3.52807496

(b) The function $W(T)$ is an increasing function. You can see this by the fact that as you read down the table in Pat (a), the net worth of the WWE goes up each time. Furthermore, you can also see in the table that the rate of change is always positive. When the rate of change is positive, the original function (in this case $W(T)$) is increasing.

(c) The WWE's net worth is a concave up function. You can see this from the table, by virtue of the fact that the rate of change increases as you read down the table. If the rate of change is increasing then the original function is concave up.

(d) There were two possible, acceptable answers here, depending on whether you thought that the independent variable, T, could be negative or not. (In practical reality, it does not make sense to allow T to be negative as T represents the number of years *since* WWE's IPO. However, if you look at $W(T)$ as a purely mathematical object with no necessary tie to reality, then $T < 0$ is certainly mathematically possible.)

Case 1: $T < 0$ **is a practical impossibility**

In this case, the function $W(T)$ will not have a horizontal asymptote. This is because you are only interested in positive T-values. As T increases, $W(T)$ is an increasing, concave up function. Therefore, the graph of $W(T)$ continues to rise, and does so more and more steeply as time goes by. In order to have a horizontal asymptote, we would need $W(T)$ to rise less and less steeply until finally, it had leveled off and closely resembled a horizontal line. However, this increasing and concave up $W(T)$ does not do that. Therefore there is no horizontal asymptote.

Case 2: $T < 0$ **is a mathematical possibility**

In this case, we note from the rate of change equation,

$$\frac{\Delta W}{\Delta T} = 0.13 \cdot W(T) - 0.194$$

that when $W(T) \approx 1.4923$, the rate of change is close to zero, indicating a horizontal asymptote. However, as $W(0) = 2.3$, this can only happen if $T < 0$. Therefore, if we allow the possibility of negative T values, then $W(T)$ can have a horizontal asymptote, although it occurs to the left of the vertical axis.

6.(a) In this problem, we are trying to find all of the x-values where the functions $f(x)$ and $g(x)$ give exactly the same output. As the function $g(x)$ is only defined at a handful of values (-4, -2, 0, 1, 2, 3) an efficient way to find all of the x values is to just compare the values of $f(x)$ and $g(x)$ at these few values. This is done in the table given below.

x	$f(x)$	$g(x)$	$f(x) = g(x)$?
-4	0	0	Yes
-2	1.5	-5	No
0	3	3	Yes
1	-1.5	-1.5	Yes
2	-1.5	0	No
3	-1.2	-3	No

Therefore, all of the values of x for which $f(x) = g(x)$ are: $x = -4$, $x = 0$ and $x = 1$.

6.(b) To find all of the values of k for which $2 \cdot f(2) - f(1) = g(k)$ you can first evaluate the left hand side of the equation, and then examine the table to see if the function $g(x)$ can ever equal that value.

From the graph supplied on the midterm,

$$2 \cdot f(2) - f(1) = 2 \cdot (-1.5) - (-1.5) = -1.5.$$

From the table, the only time that $g(x)$ has a value of -1.5 is when $x = 1$. Therefore, the only value of k that satisfies the equation $2 \cdot f(2) - f(1) = g(k)$ is $k = 1$.

6.(c) In order to be able to evaluate (hat is, to find the unique value) of $f^{-1}(-1.5) + g^{-1}(0)$, these inverses must have unique values themselves. However, if you inspect the graph given on the midterm, you will see any value of x from the interval (0, 2] satisfies the equation $f(x) = -1.5$. Likewise, both $x = -4$ and $x = 2$ satisfy the equation $g(x) = 0$. It is therefore impossible to evaluate (that is, find the unique value) of $f^{-1}(-1.5) + g^{-1}(0)$ because this collection of symbols corresponds to many different values, not one unique value.

7. Two important points to observe here is that although the input to the function (the amount of time spent underwater while scuba diving) is a time, and the output from the function (the amount of time that a diver must wait before flying) is also a time, the units of the input and output are different. The input to the function is measured in minutes whereas the output from the function is measured in hours.

(a) The amount of time before you can fly is given by: $0.8 \cdot W(t + 15)$.

The "+15" on the inside of the function reflects the extra 15 minutes underwater, and the multiplication by 0.8 on the outside of the function reflect the fact that the wait time before flying is only 80% of normal.

(b) The amount of time before you can fly is given by: $W(t - 10) + 24$.

The "-10" on the inside of the function reflects the reduction in dive time by ten minutes, and the "+24" on the outside reflects the fact that the person chose to wait an additional day (24 hours) over and above the prediction of the diving computer before flying.

(c) There are many possible answers here. In order to get full credit, a story had to have the following elements:

- A 50% reduction in the time spent underwater.
- A four-hour increase in the wait time before flying.
- Some semblance of plausibility.

For example: While diving, a large tiger shark swam into the area and surprised me. The shock of seeing such a large animal at close quarters scared me, and I decided to return to the surface despite the fact I was only halfway through my dive. I was so shaken by the

experience that I decided to medicate myself in the airport bar, missed my flight and had to wait an additional four hours until another flight was available.

(d)   Similar criteria apply here, although the story must now reflect a five-minute reduction in dive time and a doubling of the wait time.

For example: For the last year, I have had trouble with a recurring sinus infection. During my last dive, my sinuses started acting up again and eventually became so painful that I had to cut my dive short by five minutes. Acting on the advice of my doctor, I waited twice as long as normal before flying in order not to further exacerbate my condition.

8.   To find an equation for the polynomial function, $f(x)$, whose graph is pictured below, you need to do three things:

   (a)   Locate the x-coordinates of all x-intercepts (also known as "roots").
   (b)   Determine the multiplicity of each root.
   (c)   Use a point on the graph to determine the constant of proportionality, k.
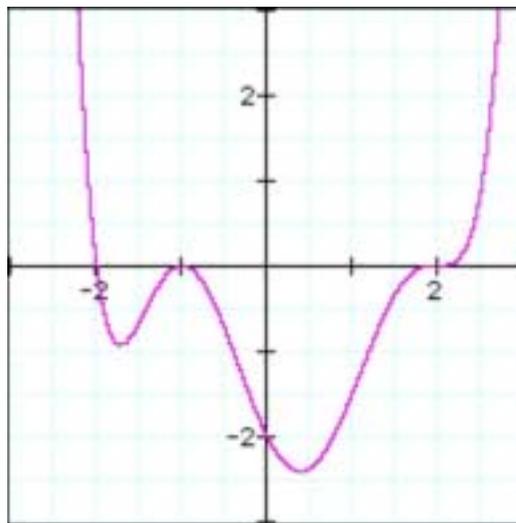
(a)   Locating the roots of the polynomial function

Inspection of the graph shows that the roots are located at $x = -2$, $x = -1$ and $x = 2$.

(b)   Determining the multiplicities of the roots

As you learned in the "Analyzing Biomedical Data" lab, you can determine the multiplicity of a root by carefully examining the shape that the graph makes as it cuts through the x-axis. The three possibilities that you have to worry about in Math Xa are multiplicity 1, 2 or 3 (see diagram below).
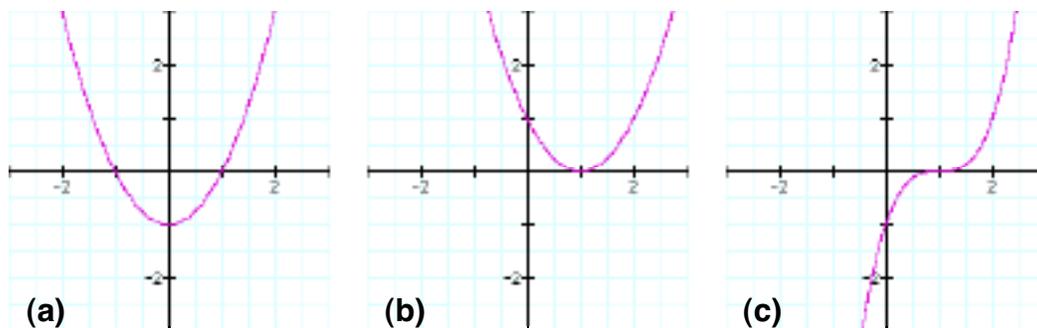
Determining the multiplicity of a root. (a) This polynomial has two roots, each with multiplicity equal to one. (b) One root with multiplicity equal to 2. (c) One root with multiplicity equal to 3.

The multiplicity of each root of the polynomial function, $f(x)$, whose equation we are trying to find is given in the table below.

Root	Multiplicity
−2	1
−1	2
2	3

(c) Put the equation together and calculate the value of the constant of proportionality

The x-coordinates and multiplicities of the roots are combined to form an equation for the polynomial function:

$$f(x) = k \cdot \left(x - (-2)\right)^1 \cdot \left(x - (-1)\right)^2 \cdot \left(x - 2\right)^3$$

where k is a number called the *constant of proportionality*.

To determine the value of the constant of proportionality, you select a point from the graph whose x- and y-coordinates you can discern very precisely, and which is not one of the roots. In this particular case, the point $(0, -2)$ is convenient. These values for x and y are now substituted into the equation for the polynomial function.

$$-2 = k \cdot \left(0 - (-2)\right)^1 \cdot \left(0 - (-1)\right)^2 \cdot \left(0 - 2\right)^3 = -16 \cdot k.$$

Solving for k gives $k = \frac{1}{8}$, and substituting this back into the equation for $f(x)$ gives the final equation for the polynomial function:

$$f(x) = \frac{1}{8} \cdot \left(x - (-2)\right)^1 \cdot \left(x - (-1)\right)^2 \cdot \left(x - 2\right)^3.$$