Exam 2: Solutions

Problem 1.

The graph of the **derivative** of the function $f(x)$ is shown below.

GRAPH OF THE DERIVATIVE
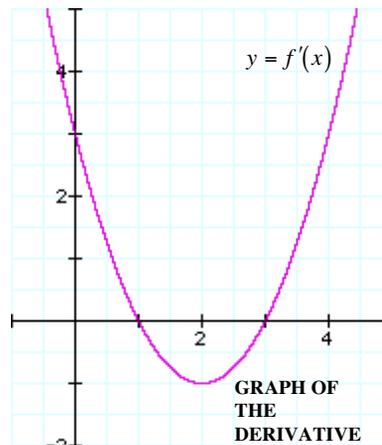
(a)   The critical points of the function $f(x)$ are located at the points where the graph of the derivative crosses the x-axis. These points are $x = 1$ and $x = 3$.

(b)   The critical points can be classified as maximums or minimums by checking the sign of the derivative (+ or −) slightly to the left and slightly to the right of the critical point. The presence of a maximum is indicated by a positive derivative on the left of the critical point and a negative derivative on the right. The opposite pattern of signs (negative on the left and positive on the right) indicates a minimum.
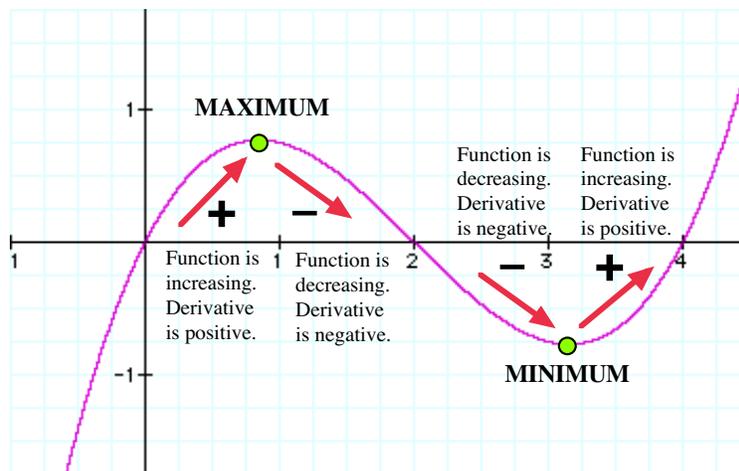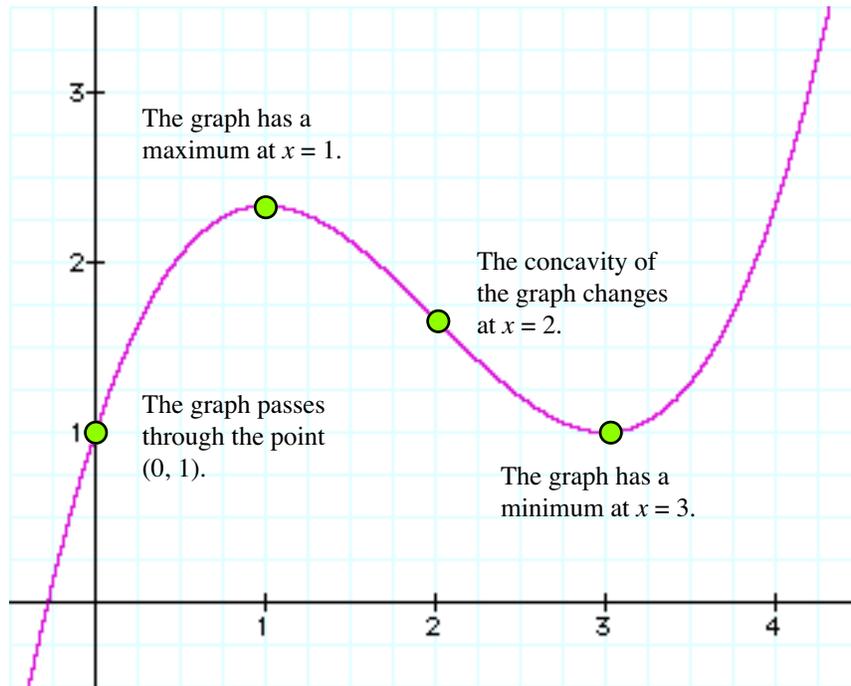
The graph of the derivative shows that the critical point at $x = 1$ is a maximum and the critical point at $x = 3$ is a minimum.

(c) Points on the graph of the original function at which the concavity changes are called inflection points. Inflection points on the graph of the original function create maximum and minimum values on the graph of the derivative. The given derivative graph has a minimum at $x = 2$, so the original function must have an inflection point at $x = 2$.

Alternatively, you can locate this point from the derivative graph by following the following reasoning:

When the derivative is decreasing, the graph of the original function $f(x)$ is concave down. When the derivative is increasing, the graph of the original function $f(x)$ is concave up. The point(s) of inflection of $f(x)$ will occur at any point where the derivative graph goes from increasing to decreasing or vice versa. If you look at the derivative graph, you will be able to see that the only point where this occurs is at $x = 2$.

(d) A graph that is compatible with all of the information in Parts (a), (b) and (c) and which goes through the point (0, 1) is shown below. There are many possible answers to this problem. The main thing that we were looking for was that you drew a graph that was consistent with the information about the original function that you had inferred from the derivative graph.

The graph has a maximum at $x = 1$.

The concavity of the graph changes at $x = 2$.

The graph passes through the point (0, 1).

The graph has a minimum at $x = 3$.

Problem 2.

In this problem the function $g(x)$ will always refer to the function defined by the equation:

$$g(x) = \frac{f(x)}{x}.$$

All that you could assume about the function $f(x)$ was that $f(x)$ has a derivative and that:

- $f(1) = 1$
- $f(2) = 3$

- $f'(1) = 1$
- $f'(2) = \frac{1}{2}$.

To find the derivative of $g(x)$, it is appropriate to use the Quotient Rule for derivatives. The Quotient rule gives the following formula for the derivative of $g(x)$.

$$g'(x) = \frac{f'(x) \cdot x - 1 \cdot f(x)}{x^2}$$

(a) To calculate $g'(1)$ will substitute the given function and derivative values into the expression above. This gives the following.

$$g'(1) = \frac{f'(1) \cdot 1 - 1 \cdot f(1)}{1^2} = 0$$

(b) To calculate $g'(2)$ will substitute the given function and derivative values into the expression above. This gives the following.

$$g'(2) = \frac{f'(2) \cdot 2 - 1 \cdot f(2)}{2^2} = \frac{-1}{2}$$

(c) To calculate the derivative of $m(x)$ it is appropriate to use the product rule for derivatives. The Product rule gives the following formula for the derivative of $m(x)$.

$$m'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Substituting the function and derivative values given above (including the value of $g'(2)$ from Part (b)) gives the following.

$$m'(2) = f'(2) \cdot g(2) + f(2) \cdot g'(2) = \frac{-3}{4}$$

(d) There are several ways to approach this calculation. One of the most direct is to recognize that the definition of $g(x)$ simplifies the formula for $k(x)$ quite a lot.

$$k(x) = \frac{g(x)}{f(x)} = \frac{1}{f(x)} \cdot g(x) = \frac{1}{f(x)} \cdot \frac{f(x)}{x} = \frac{1}{x}.$$

Using the Power rule for derivatives gives that $k'(x) = -1 \cdot x^{-2}$. Substituting $x = 2$ into this derivative formula gives that $k'(x) = \frac{-1}{4}$.

Problem 3.

In this problem, the function $f(x)$ will always refer to the function defined by:

$$f(x) = \frac{1}{x}.$$

(a) Using the power rule for derivatives, the derivative is: $f'(x) = -1 \cdot x^{-2}$.

(b) Using the formula given for $f(x)$ to create the difference quotient gives the following.

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}.$$

(c) When simplifying the difference quotient, your objective is always to reduce the difference quotient to the point where you are able to cancel out the h that appears in the denominator. This is because you will eventually have to take the limit as $h \to 0$, and having a factor of h left on the bottom of the fraction will make the limit calculation unnecessarily complicated.

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \frac{\dfrac{x - (x+h)}{x \cdot (x+h)}}{h} = \frac{-h}{h \cdot x \cdot (x+h)}.$$

You can now cancel the h's to remove the factor of h in the denominator of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x \cdot (x+h)}.$$

(d) The derivative is equal to the limit of the difference quotient as $h \to 0$. Taking the limit of the expression obtained in Part (c) of this problem gives the following.
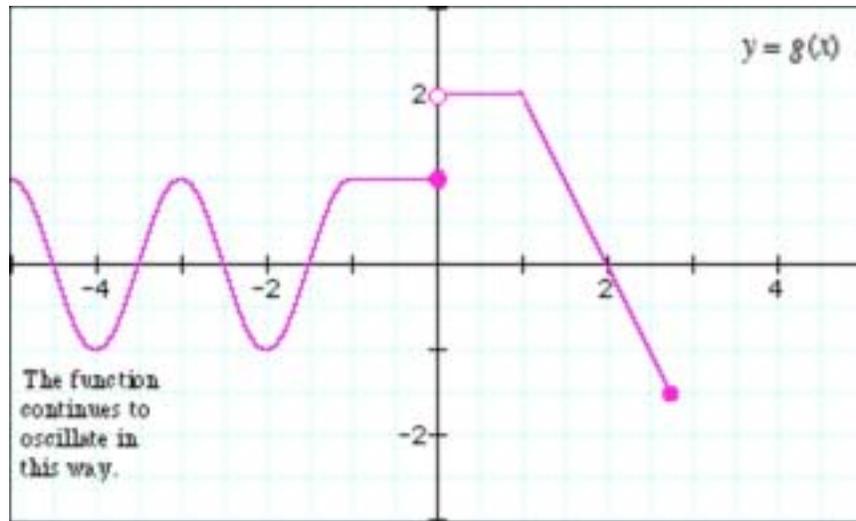
$$f'(x) = \frac{\text{Lim}}{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\text{Lim}}{h \to 0} \frac{-1}{x \cdot (x+h)} = \frac{-1}{x^2}.$$

Problem 4:

In this problem, the function f will always refer to the function defined by the equation:

$$f(x) = \frac{1-x}{1-x^2}$$

and the function g will always refer to the function defined by the graph:

$y = g(x)$

The function continues to oscillate in this way.

(a) $\displaystyle \lim_{x \to 1} f(x) = \frac{1}{2}.$

To deduce this limit, note that the denominator of f can be factored:

$$f(x) = \frac{1-x}{1-x^2} = \frac{1-x}{(1-x)\cdot(1+x)} = \frac{1-x}{1-x} \cdot \frac{1}{1+x}.$$

When x is close to 1, $f(x)$ will be very close to $(1)\cdot(1/2) = 1/2$.

(b) $\displaystyle \lim_{x \to -1^+} f(x) = +\infty.$

To deduce this limit, note that the denominator of f can be factored:

$$f(x) = \frac{1-x}{1-x^2} = \frac{1-x}{(1-x)\cdot(1+x)} = \frac{1-x}{1-x} \cdot \frac{1}{1+x}.$$

When x is slightly smaller than -1, $f(x)$ will look like:

$$f(x) = \frac{1-x}{1-x} \cdot \frac{1}{1+x} \approx \frac{(\text{Approximately } 2)}{(\text{Approximately } 2)} \cdot \frac{1}{(\text{Very small negative})} = \text{Very big negative}.$$

(c) $\displaystyle\lim_{x \to 0} g(x)$ does not exist because the left and right hand limits of $g(x)$ at $x = 0$ are not equal. From the graph of $g(x)$ that was given, you can see that the left hand limit is equal to $+1$ whereas the right hand limit is equal to $+2$.

(d) $\displaystyle\lim_{x \to 2} \frac{f(x)}{g(x)}$ does not exist because the left and right hand limits of this quotient are not equal at $x = 2$. The left hand limit is equal to $+\infty$ whereas the right hand limit is equal to $-\infty$. We will give the analysis for the right hand limit in complete detail below; the determination of the left hand limit can be completed via a very similar analysis.

To deduce the value of the right hand limit, you can look at the behavior of the functions $f(x)$ and $g(x)$ as each approach $x = 2$ from the right, and then combine these to understand how the quotient $\frac{f(x)}{g(x)}$ behaves as x approaches 2 from the right.

Behavior of $f(x)$ as x approaches 2 from the right.

The function f is defined when $x = 2$. Its value (which is equal to the limit from the right) is: $f(2) = 1/3$. Therefore, when x is a number slightly larger than 2, $f(x)$ will be approximately equal to 1/3.

Behavior of $g(x)$ as x approaches 2 from the right.

Inspection of the graph of g shows that when x is a number slightly larger than 2, $g(x)$ is a negative number that is very, very close to zero.

Behavior of $\frac{f(x)}{g(x)}$ as x approaches 2 from the right.

Combining the previous two observations, you have that when x is a number slightly greater than 2,

$$\frac{f(x)}{g(x)} \approx \frac{\frac{1}{3}}{(\text{Small negative})} = \text{Large negative}.$$

So, as x gets closer and closer to 2 from the right, $\frac{f(x)}{g(x)}$ is a larger and larger negative number.

Problem 5.

The population of Hungary is represented by the equation:

$$P(t) = 10.8 \cdot (0.998)^T,$$

where P will always refer to the population of Hungary in units of millions of people and T will always refer to the number of years since 1990.

(a) Since $P(T)$ is an exponential function, you use the rule for differentiating exponential functions:

$$\text{If: } f(x) = A \cdot B^x \text{ then: } f'(x) = A \cdot B^x \cdot \ln(B).$$

This gives that:
$$P'(T) = 10.8 \cdot (0.998)^T \cdot \ln(0.998) \approx -0.0216 \cdot (0.998)^T.$$

(b) Evaluating the equation from Part (a) for $T = 11$ gives:

$$P'(11) = -0.0216 \cdot (0.998)^{11} \approx -0.02115.$$

A practical interpretation of this number is that when T increases from $T = 11$ to $T = 12$, then the population of Hungary will decrease by approximately 0.02115 million people (i.e. about 21,150 people).

(c) Your objective in this part is to find the value of T that gives $P(T) = 5.8$. To do this you could set up the equation:

$$5.8 = 10.8 \cdot (0.998)^T.$$

Simplifying gives:

$$(0.998)^T = \frac{5.8}{10.8}.$$

Taking logs of both sides and re-arranging to make T the subject of the equation gives:
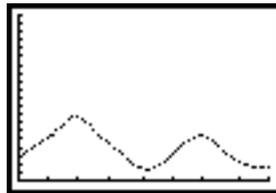
$$T \cdot \log(0.998) = \log\left(\frac{5.8}{10.8}\right)$$

$$T = \frac{\log\left(\frac{5.8}{10.8}\right)}{\log(0.998)} \approx 310.53.$$

So, in the year (1990 + 310) = 2300, the population of Hungary will reach 5.8 million people.

Problem 6.

The graph shown below is the distance-time graph obtained by a group of Math Xa students using a calculator and a motion detector.


(a)   A velocity-time graph that is consistent with the distance-time graph shown above is given below. There are many possible answers here, but anything that is consistent with the distance time graph given is fine.
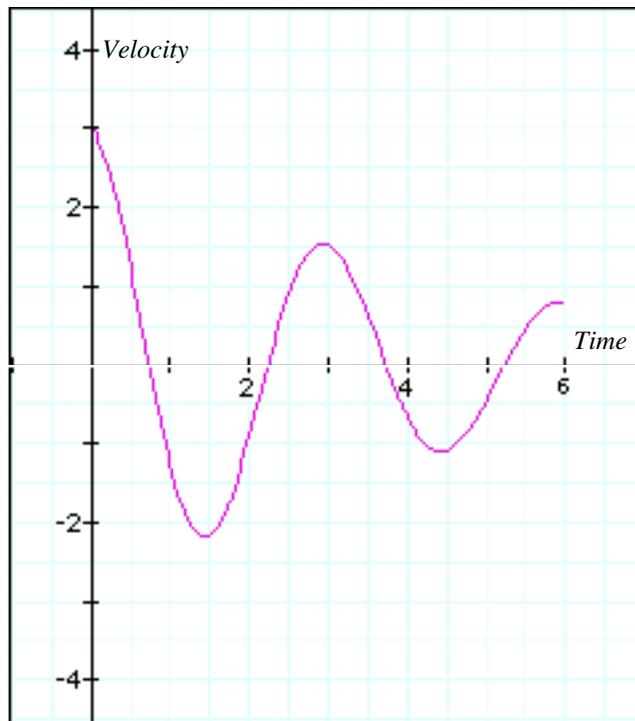
(b)   The *x*-intercepts on the velocity-time graph correspond to the maximum and minimum values on the distance-time graph.

(c) The intervals for the velocity-time graph shown above are [0.75, 2.25] and [3.75, 5.25]. If you used the horizontal axes provided on the actual exam, then your answers will probably more closely resemble [2, 4] and [6, 8]. During these time intervals, the Math Xa student who made the graph was walking towards the motion detector.

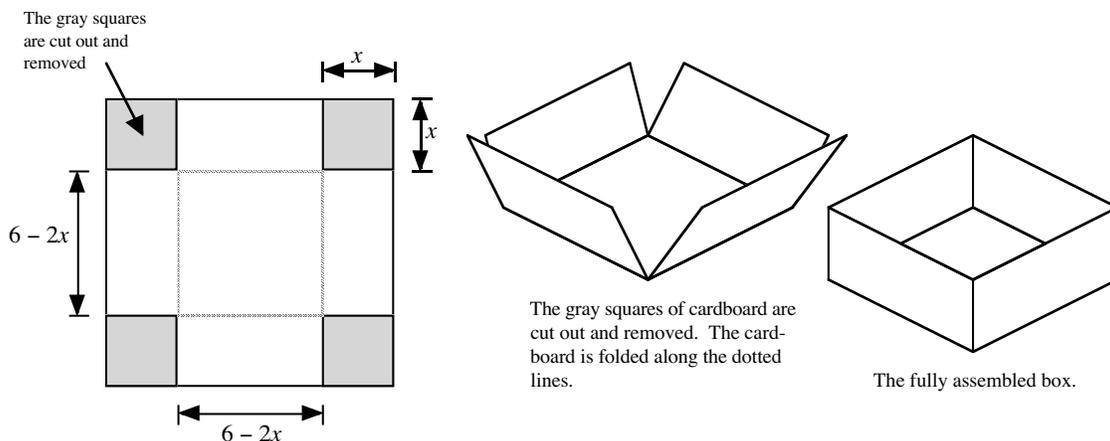**(d)** The discontinuous distance-time graph that the students produced is shown below.
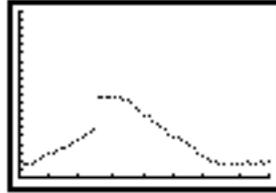
The most readily feasible way to produce this graph is to have two people work together. At the beginning, one person (Alan) stands close to the motion detector while the second person (Beverly) stands further away. When the motion detector is started, Alan walks away from it with a steady speed. Beverly stays where she is. After about two seconds, Alan leaps to one side, out of the view of the motion detector. At this time, Beverly begins to walk towards the motion detector. She walks very slowly at first, but gains speed as she moves closer to the detector.
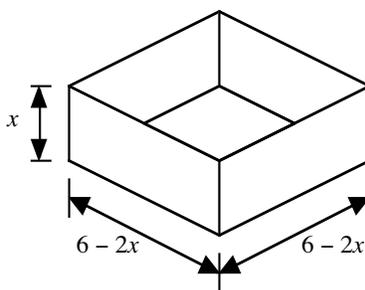
Problem 7.

(a) $y = \dfrac{3 \cdot (x + 1) \cdot (x - 2)}{x}$.   **CORRESPONDING GRAPH = II**

(b) $y = \dfrac{3 \cdot (x + 1) \cdot (x - 1)}{(x - 2) \cdot (x + 2)}$.   **CORRESPONDING GRAPH = V**

(c) $y = \dfrac{3 \cdot (x + 1) \cdot (x - 2)}{x^2}$.   **CORRESPONDING GRAPH = IV**

(d) $y = \dfrac{5 \cdot (x - 1)}{(x + 1) \cdot (x + 2)}$.   **CORRESPONDING GRAPH = III**

Problem 8.

In this problem, a box is formed from a square piece of cardboard. Small squares (each with length x) are cut out of the corners and the sides are folded up to form the box.

The gray squares are cut out and removed

The gray squares of cardboard are cut out and removed. The cardboard is folded along the dotted lines.

The fully assembled box.

(a) When the box is folded up, it will resemble the diagram shown below.

The volume of the box (once it has been folded up) will be the product of length, width and height. In terms of x the volume will be given by the following formula.

$$V(x) = x \cdot (6 - 2x)^2 = 36 \cdot x - 24 \cdot x^2 + 4 \cdot x^3.$$

(b) The derivative of the volume formula is shown below.

$$V'(x) = 36 - 48x + 12x^2 = 12(x - 1)(x - 3).$$

(c) The derivative is equal to zero when $x = 1$ or $x = 3$. Which one of these gives maximum volume? The second point, $x = 3$, gives zero volume so that can't be it. The maximum volume is obtained using $x = 1$. You can confirm this for certain by checking the value of the derivative $V'(x)$ at a point slightly to the left of $x = 1$ and at a point slightly to the right of $x = 1$. The results of doing this are shown in the table below, and the pattern of the derivative signs confirm that $x = 1$ does indeed give maximum volume.

x	0.9	1	1.1
$V'(x)$	+2.52	0	−2.28