Practice Problems: Test #1, Set #1

Important Information:

1. The first test will be held on **Wednesday October 30 from 7-9pm in Science Center D**.

2. The test will include approximately eight problems (each with multiple parts).

3. You will have 2 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test.

5. The specific topics that will be tested are:

   - The definition of a function.
   - Representations of functions (graphs, numbers/tables, written descriptions, equations).
   - Interpreting graphs, tables, words and symbols.
   - Modeling relationships using linear functions.
   - Interpreting the parameters of a linear function.
   - Calculating and interpreting rates of change.
   - Exponential growth and exponential functions.
   - Modeling relationships using exponential functions.
   - Solving equations involving linear and exponential functions (logarithms).
   - Rates of change and concavity.
   - Approximating functions using their rates of change (Euler’s method).
   - Transformations of functions (shifts, stretches and reflections of functions).
   - Power functions.
   - Predicting the appearance of the graph of a power function.
   - Modeling relationships using power functions.
   - Polynomial functions.
   - Predicting the appearance of the graph of a polynomial, exponential or power function.
   - Fitting a polynomial function to data.
   - Compositions of functions.
   - The concept of the inverse of a function.
   - Functions defined in pieces.

6. The problems included here have been chosen because they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

7. Good places to go for help include:

   - Office hours.
   - The labs on Tuesday 10/29.
   - The Math Question Center
   - The course-wide review on Tuesday 10/29 from 5:00-7:00pm in Science Center D.

8. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

1. In this question, the function $h(x)$ will always refer to the function defined in Figure 1 (below).

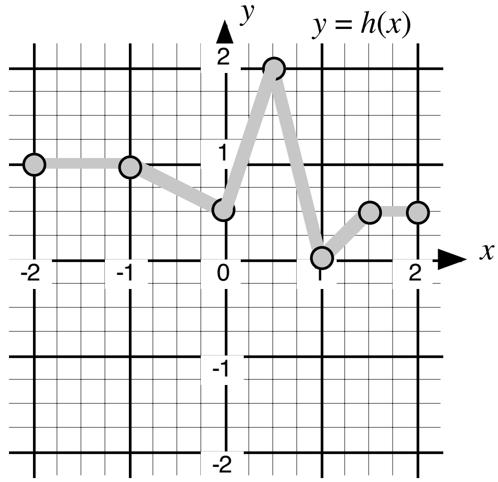

Figure 1: Graph of $y = h(x)$ for Questions 1 and 2.

(a) A new function, p is defined by the equation:

$$p(x) = -h(x + 1) + 2.$$

Describe how $p(x)$ can be obtained from the function $h(x)$ using a series of geometrical transformations. (You should be careful to describe both the transformations that must be performed as well as the order that they should be performed in.)

(b) Sketch an accurate graph of $y = p(x)$ labeling all of the points that lie at the end of a line segment.

2. Figure 2[1] shows the readout from an electrocardiogram (ECG) machine. The signal shown in Figure 2 depicts the normal heart function of an individual at rest. The horizontal direction represents time, with each small square representing 0.04 seconds. The vertical axis represents electrical activity, with each small square representing a voltage of 0.1 mV.

Figure 2: Output from an ECG machine showing a normal heart beat (i.e. a sinus rhythm).

(a) Use the graph shown in Figure 2 to calculate the length (in seconds) of each heartbeat. (One complete heartbeat consists of a complete set of pulses together with one of the long, flat portions of the graph.) How many times does this person's heart beat each minute?

---

[1] Image source: http://rnbob.members.tripod.com/

(b) The ECG output shown in Figure 2 was from a 31 year old, healthy male. Let the function B be the function graphed in Figure 2. That is, Figure 2 shows the graph of $y = B(t)$. According to some physiological models[2], the maximum heart rate for a healthy 31 year-old male is 189 beats per minute. Modify the function B to create a new function that will represent the ECG trace when the person's heart is beating at the maximum possible rate. (Note: you do **not** have to find an explicit equation for $B(t)$.)

(c) Each individual heartbeat includes an upward and a downward "spike." In cardiology, the large upward spike is called the "R" wave and the smaller downward spike is called the "S" wave. Working from Figure 2, determine the difference in voltage between the top of the "R" wave and the bottom of the "S" wave. How could you modify the function B so that it would describe the ECG of a patient with a 30 mV difference between the "R" and "S" waves? (Note: you do **not** have to find an explicit equation for $B(t)$.)

3. Sometimes, the inverse of a function is a function in its own right. Suppose that you have a function f and you wish to know whether the inverse is a function in its own right. The usual test is called the **Horizontal Line Test**. To perform the horizontal line test, you plot a graph of $y = f(x)$.

- If all horizontal lines cut the graph of $y = f(x)$ in one and only one place, then the inverse is a function in its own right.

- If one or more horizontal lines cut the graph of $y = f(x)$ in more than one place, then the inverse is not a function in its own right.

(a) Use the horizontal line test to decide whether or not the inverses of any of the functions shown in Figure 3 (below) are functions in their own right.

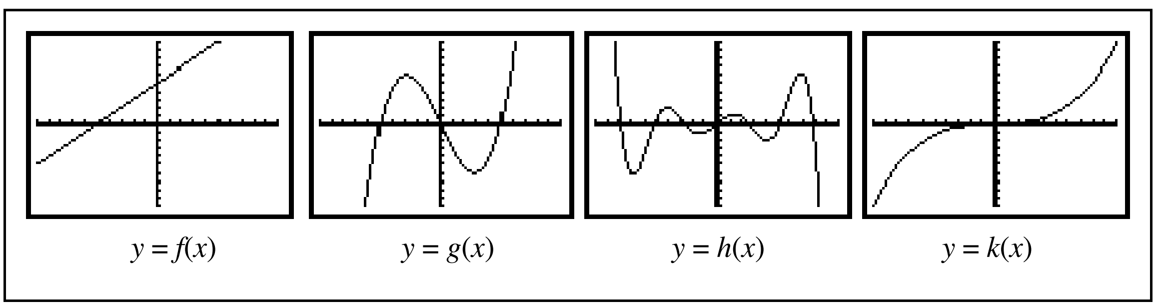

Figure 3: Functions and graphs for Question 3(a).

(b) The graph shown in Figure 4 (over) shows a graph of pH versus time. The data shown in the graph were collected using an electronic pH meter and a tank containing ordinary Boston drinking water[3]. The water was slowly acidified by bubbling carbon dioxide gas though it. The pH of the water is a function of time. Explain how you can deduce this from Figure 4. Is the inverse a function in its own right? Explain why or why not.

(c) Use the graph from Figure 4 to sketch a graph that shows pH as the independent variable and time as the dependent variable. What is the geometrical relationship between the curve you have sketched and the curve shown in Figure 4?

---

[2] Source: http://my.webmd.com/heartrate

[3] According to the Massachusetts Regional Water Authority, the average pH of Boston tap water is 9.0

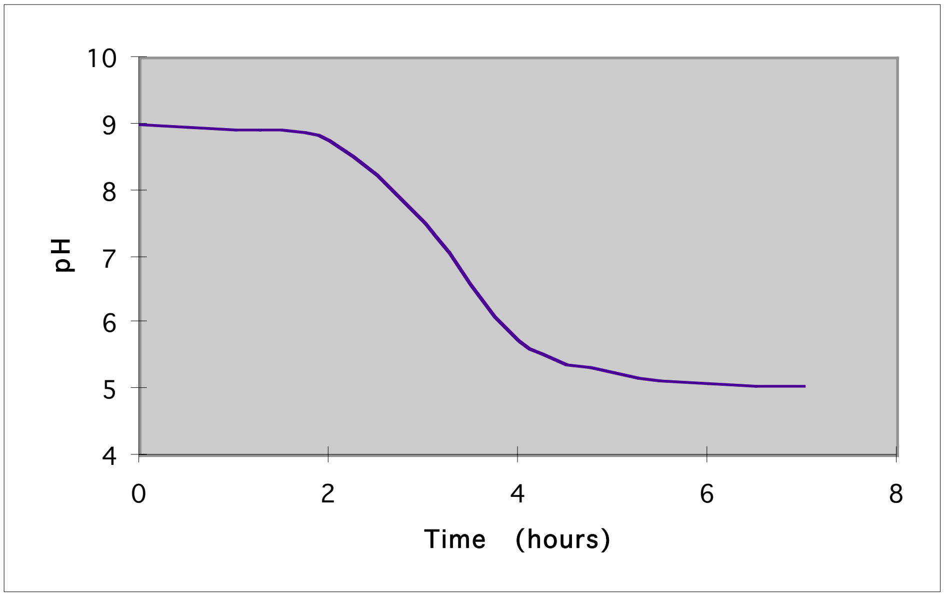

Figure 4: Titration curve for Boston drinking water acidified by CO_2 gas.

(d) A function $f(x)$ is defined by the equation given below. The inverse of $f(x)$ is a function in its own right. (You do not need to prove that this is the case.) Find an equation for the inverse of $f(x)$.

$$f(x) = \frac{1+x}{1-x}.$$

4. In the last twenty years, tuition and fee costs at four year colleges have continued to rise. Table 1 (below) shows the average yearly tuition costs for private and public colleges from 1971 to 1999[4].

Academic Year	Tuition - Private (Dollars)	Tuition - Public (Dollars)
1971-1972	1820	376
1980-1981	3617	804
1984-1985	5556	1228
1989-1990	8663	1696
1998-1999	14508	3243

Table 1: Average Cost of Tuition and Fees, 4-Year Colleges.

(a) Find an equation (from the choice of linear, exponential or power) that does the best job of representing the cost of tuition at a private university as a function of time. Describe the criteria you used to decide which is best.

(b) Find an equation (from the choice of linear, exponential or power) that does the best job of representing the cost of tuition at a public university as a function of time. Describe the criteria you used to decide which is best.

(c) Calculate the total cost of tuition for students in the following situations:

[4] Source: *The College Board*, 2000.

Student A: Planned to attend a private university starting in Fall 2000 and graduate in four years.

Student B: Plans to attend a public university starting in Fall 2025 and graduate in four years.

Student C: Plans to attend a private university starting in Fall 2025 and graduate in four years.

5. A basic diving computer (see Figure 5[5]) incorporates a depth gauge and a clock. The computer measures the depth of the dive, and the amount of time that the person remains underwater.

Figure 5: Wrist mounted diving computer manufactured by UWATEC. (a) The dive computer unit. (b) During a dive the computer displays current depth (62 feet), maximum depth (65 feet), time under (3 minutes) and time remaining (46 minutes). (c) After the dive the computer displays the number of hours before the diver can fly (9 hours) and the time required to completely eliminate dissolved nitrogen from the blood (28 hours, 29 minutes).

Diving computers have become popular among SCUBA enthusiasts as they can help divers to avoid painful and life-threatening "decompression sickness" (also called "DCS" or "the bends"). DCS can occur when a diver swims to the surface too quickly after a deep dive, or when a diver flies in an airplane too soon after a dive. To help prevent DCS, the Divers Alert Network[6] suggests a minimum wait of 12 hours between a single dive and flying, and a 24 hour wait between multiple dives and flying. Similarly, the U.S. Coast Guard[7] recommends a wait of at least twelve hours between diving and flying.

If you always dive to the same depth (e.g. 65 feet) then the amount of time that a diving computer will tell you to wait between diving and flying depends on how much time you spent underwater. For example, if you spent t minutes underwater, you could express the number of hours that the computer thinks you should wait as $W(t)$.

In (a)-(c), express the number of hours that you will have to wait as a symbolic statement using t, $W(t)$, numbers, +, -, *, ÷ etc. Remember, you are looking for an expression that involves function notation, not a formula or a number.

(a) I was supposed to be underwater for t minutes, but a large tiger shark appeared in the area. I had to hide on the bottom for an extra fifteen minutes until the shark swam away.

(b) I know that my body takes longer to adapt than average, so if I spend t minutes underwater, my actual wait time will be 3 hours longer than the prediction of the computer.

---

[5] Image source: http://www.uwatec.com/

[6] Source: http://www.dan.ycg.org/

[7] Source: U.S. Coast Guard Boat Crew Seamanship Manual, 1998. (COMDTINST M16798.27)

(c) I was supposed to spend t minutes underwater but cut it short by five minutes. As I was breathing a special blend of gases instead of compressed air, my wait time was two hours less than the prediction of the computer.

(d) Suppose that the computer prediction of your wait time is M hours. Express the number of minutes that you spent underwater as a collection of symbols such as t, $W(t)$, numbers, +, -, *, ÷ etc.

6. In this problem, the two functions $f(x)$ and $g(x)$ are defined by the graphs shown below. The x-intercept of f is located at $x = 2$. The x-intercepts of g are located at $x = 1$ and $x = 3$. As shown in Figure 6, the domain of the function f is the interval [−2, 3], and the domain of the function g is the interval [0, 4].

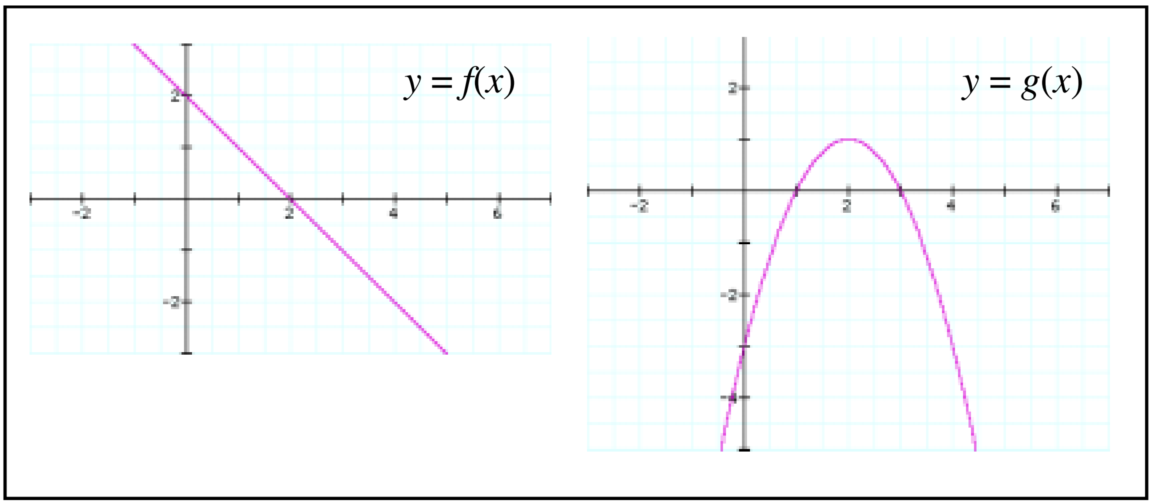

Figure 6: Graphs of the functions $f(x)$ and $g(x)$ for Question 6.

(a) Using a suitable set of axes, plot a graph showing:

$$y = g(f(x))$$

concentrating on the interval between $x = -1$ and $x = +1$.

(b) Using a suitable set of axes, plot a graph showing:

$$y = f(x) - g(x).$$

As part of your answer, describe the set of x-values that you can assign y-values to when sketching the graph of $y = f(x) - g(x)$.

(c) Consider two new functions, h and p, defined by the equations:

$$h(x) = \frac{f(x)}{g(x)} \quad \text{and} \quad p(x) = \frac{g(x)}{f(x)}.$$

What are the domains of the functions h and p? In each case, briefly explain why some points from the interval [−2, 4] are excluded from the domain of each function.

(d) Using a suitable set of axes, plot a graph showing:

$$y = h(x)$$

concentrating on the interval between $x = 0$ and $x = 3$.

8. Table 2 (below) shows the percentage of people in the US who lived in rural areas between 1830 and 1960[8].

Year	1830	1840	1850	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950	1960
% rural	91.2	89.2	84.7	80.2	74.3	71.8	64.9	60.4	54.4	48.8	43.9	43.5	36.0	30.1

Table 2: Percentage of US population living in rural areas 1830-1960.

(a) Plot a graph showing the percentage of people living in rural areas versus year. What features would a function (increasing/decreasing, concave up/down or no concavity) need to show in order to do a reasonable job of representing the trend(s) in your plot?

(b) Find an equation that you can use to predict the percentage of people living in rural areas, given the year. Over what period(s) of time do you expect the predictions of your function to closely match the actual values of the data from Table 2?

(c) Use your equation to predict the percentage if people living in rural areas in 2050. Does your answer make sense? Explain why or why not.

9. The "mega-tooth" shark (*Carcharodon megalodon*) is a giant shark that appears to have lived between 10 and 50 million years ago[9]. Much of what we know about this shark comes from fossilized teeth (see Figures 7 and 8[10]) that have been found in coastal regions of Virginia, North Carolina, South Carolina, Georgia and Florida. Based on the size of these teeth, many scientists believe that *C. megalodon* was approximately the same size as a Greyhound bus[11] (see Figure 9[12]).

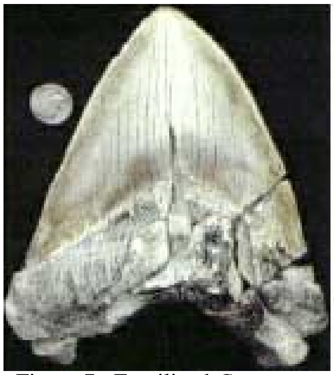

Figure 7: Fossilized *C. megalodon* tooth. (The small circle is a dime.)

Figure 8: Replica of a *C. megalodon* jaw. (The replica is about seven and a half feet tall.)

One technique for dating fossils is potassium-40 radioisotope dating. In this technique, a sample of rock is chemically analyzed and the percentage of the rock that is potassium-40 determined. Typically for fresh sediment, 2.4% of the sediment is potassium-40. Potassium-40 is a radioactive isotope (half life = 1260 million years) of potassium that decays into the inert gas argon-40.

---

[8] Source: US Bureau of the Census, *The Statistical History of the United States* (1976).
[9] Ellis, R. and McCosker, J. Eds. 1995. *Great White Shark*. Palo Alto, CA: Stanford University Press.
[10] Image source for Figures 1 and 2: http://sharksteeth.com
[11] Gottfried, M.D., L.J.V. Campagno & S.C. Bowman. 1996. Size and skeletal anatomy of the giant "Megatooth" shark *Carcharodon megalodon*. In A.P. Klimley and D.G. Ainley. Eds. *Great White Sharks: The Biology of* Carcharodon carcharias. San Diego, CA: Academic Press.
[12] Image source: http://hometown.aol.com/rjravalli/index5.html

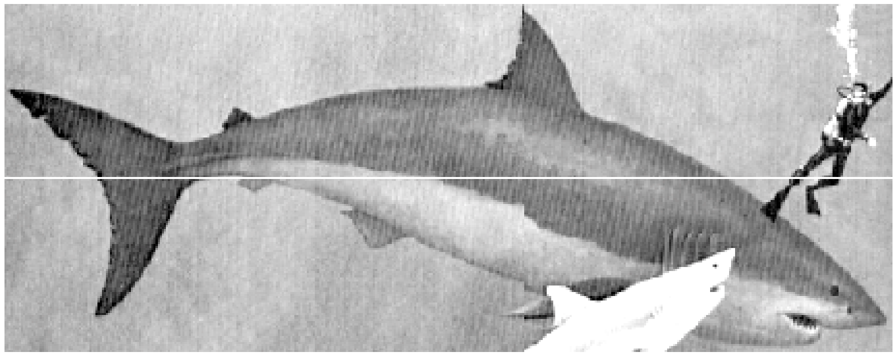

Figure 9: Artist's impression of *C. megalodon*. A diver and a modern Great White shark are also shown.

(a) What sort of function will do a good job of representing this situation? (Use the age of the rock as the independent variable and the percentage of potassium-40 as the dependent variable.) Be careful to explain your reasoning.

(b) Find an equation that will give the percentage of a rock sample that is potassium-40 as a function of the age of the rock sample.

(c) A *C. megalodon* tooth was found in a quarry in North Carolina. Lab tests determined that 2.38683% of the sediment that the tooth was found was potassium-40. About how old does that make the tooth?

10. Your answer to problem 9suggests that *C. megalodon* lived in the distant past. However, some people who are reasonably familiar with marine life (such as professional fishermen, for example) claim to have seen gigantic sharks relatively recently. One of the most credible reports was from the noted author of western novels Zane Grey. Fishing off the New Zealand coast in 1928, Grey reported seeing an "...enormous yellow and green shark with a square head, immense pectoral fins and a few white spots[13]." The creature that Grey saw is almost certainly a whale shark (*Rhincodon typus*). However, other sightings are not so easy to dismiss.

In 1959, Dr. W. Tschernezky published an analysis[14] of *C. megalodon* teeth based on the amount of manganese dioxide deposited on the teeth. Two teeth were analyzed and the thickness of the layers of MnO_2 were 1.7mm and 3.7mm thick. Under typical conditions, MnO_2 is deposited on surfaces at a constant rate of somewhere between 0.15 and 1.4 mm per thousand years[15].

(a) What sort of function would be appropriate to use if you were interested in describing the relationship between the thickness of MnO_2 and the age of the tooth? Briefly explain what clues you used to deduce that this type of function would be the most appropriate.

(b) Find two equations relating the thickness of MnO_2 to the age of the tooth: one that gives the thickness of MnO_2 on a tooth if MnO_2 is deposited at a rate of 0.15 mm per 1000 years, and a second equation that gives the thickness of MnO_2 on a tooth if MnO_2 is deposited at a rate of 1.4 mm per 1000 years.

(c) Use your equations (along with the data for the two *C. megalodon* teeth analyzed by Dr. W. Tschernezky) to find a *range* of possible ages for each of the two megalodon teeth.

---

[13] Roesch, B.S. 1998. A critical evaluation of the supposed contemporary existence of *Carcharodon megalodon*. *The Cryptozoology Review*, **3**(2): 14-24.
[14] Tschernezky, W. (1959) Age of *Carcharodon megalodon*? *Nature*, 184: 1331-1332.
[15] Roesch, B.S. 1998. A critical evaluation of the supposed contemporary existence of *Carcharodon megalodon*. *The Cryptozoology Review*, **3**(2): 14-24.