Solutions: Test #1, Set #1

These answers are provided to give you something to check your answers against and to give you some idea of how the problems were solved. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.

1.(a) The key to deciphering the transformations that were performed to create the new function p from the given function h is to start from the inside and work your way out.

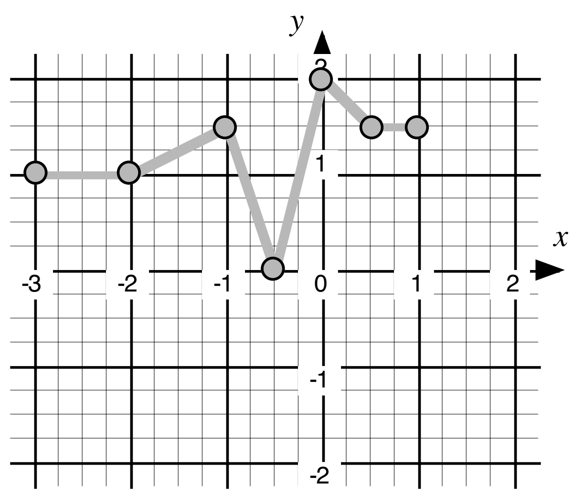
$$p(x) = -h(x + 1) + 2.$$

First transformation: Horizontal shift one unit to the left.

Second transformation: Reflection across the x-axis.

Third transformation: Vertical shift upwards by two units.

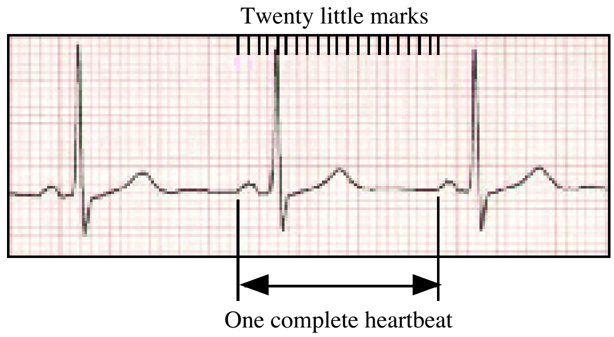
1.(b) A graph of $y = p(x)$ is shown below.



2.(a) To determine the length of time that each heart beat takes, you can:

- Count the number of squares covered by one heart beat, and,
- Multiply the number of squares by 0.04 (as each little square represents 0.04 seconds).

The length of one complete heartbeat is shown below. The little black marks on this diagram show where the edges of the squares are. Counting these from Figure 2, one heart beat covers 20 squares, so one heart beat takes $20 \times 0.04 = 0.8$ seconds.



In one minute, this person's heart will beat a total of $\dfrac{60}{0.8} = 75$ times.

2.(b) The ECG readout can be interpreted as the graph of a function $y = B(t)$. The main modification that is required is that the function be changed to show 189 heart beats in one minute, rather than just 75. This means that instead of having a duration of 0.8 seconds, each heart beat must now have a duration of:

$$\frac{60}{189} = 0.3175 \text{ seconds.}$$

The modification of the function B that will achieve this is a horizontal stretch. The appropriate stretch factor is:

$$\frac{189}{75} = \frac{0.8}{0.3175} = 2.52\,.$$

So, when the person's heart is beating at 189 beats per minute the ECG would be represented by the graph of:

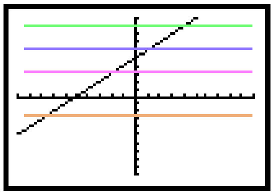
$$y = B(2.52 \cdot t).$$

2.(c) From the diagram in Part (a), the difference between the peak of the "R" wave and the bottom of the "S" wave is 20 squares. Vertically, each square represents 0.1 mV, so the difference between the top of the "R" wave and the bottom of the "S" wave in the above diagram is $20 \times 0.1 = 2$ mV.

An individual who had a difference between the top of the "R" wave and the bottom of the "S" wave of 30 mV would have an ECG that was 15 times taller than the one shown in the diagram from Part (a). The appropriate modification of $y = B(t)$ to represent this situation would be a vertical stretch with a stretch factor of 15. In other words, such an ECG would be represented by the graph of:

$$y = 15 \cdot B(t).$$

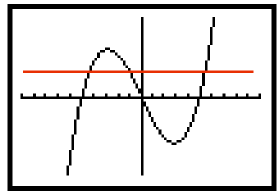
3.(a) The four functions are analyzed below.



$y = f(x)$

Any horizontal line that you draw cuts the graph in only one place.

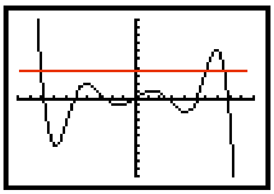
INVERSE IS A FUNCTION IN ITS OWN RIGHT



$y = g(x)$

The red line shown cuts the graph in more than one place.

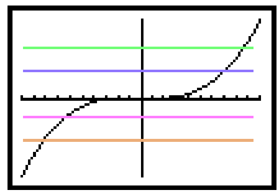
INVERSE IS NOT A FUNCTION IN ITS OWN RIGHT



$y = h(x)$

The red line shown cuts the graph in more than one place.

INVERSE IS NOT A FUNCTION IN ITS OWN RIGHT

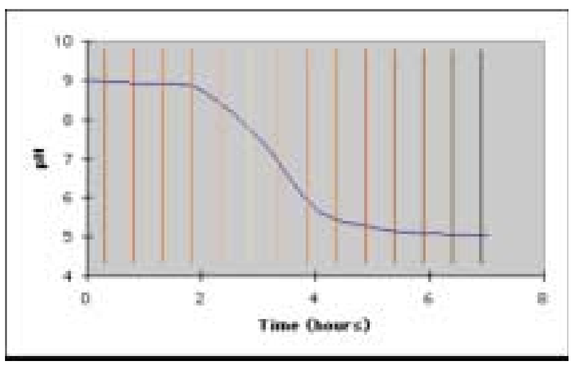


$y = k(x)$

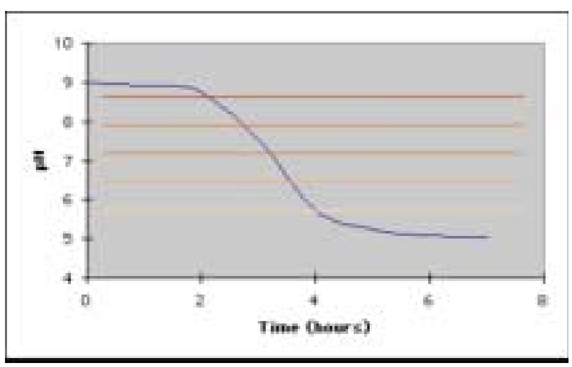
Any horizontal line that you draw cuts the graph in only one place.

INVERSE IS A FUNCTION IN ITS OWN RIGHT

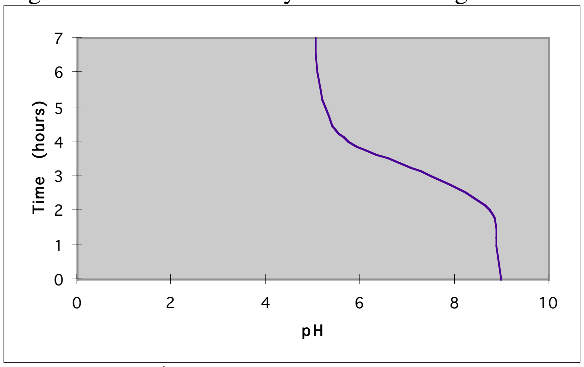
3.(b) The main requirement for a graph to be the graph of a function is that it passes the **vertical line test**. That is, no vertical line cuts the graph in more than one place. As shown in the diagram below, the graph of pH versus time meets this criteria. So, pH is a function of time.



The inverse of the function (that is, time as a function of pH) is a function in its own right. This can be deduced from the pH versus time graph by checking to see whether or not it passes the **horizontal line test**. From the diagram shown below, it does appear to be the case that any horizontal line that you could draw will only cut the graph in one place.



3.(c) The diagram below shows the plot of time versus pH. The geometrical relationship between this graph and the graph of the original function is that they are mirror images in the line $y = x$.



3.(d) In order to find an equation for $f^{-1}(x)$ you can:

- Replace each x in the equation of the function by $f^{-1}(x)$,
- Replace $f(x)$ by x, and then,
- Re-arrange to make $f^{-1}(x)$ the subject.

Carrying out this calculation:

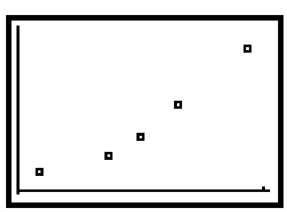
$$x = \frac{1 + f^{-1}(x)}{1 - f^{-1}(x)}$$

$$x \cdot \left[1 - f^{-1}(x)\right] = 1 + f^{-1}(x)$$ (Multiply both sides by denominator)

$$x - 1 = f^{-1}(x) + x \cdot f^{-1}(x)$$ (Get terms with inverse in them on one side of equation, terms without inverse on the other)

$$f^{-1}(x) = \frac{x - 1}{x + 1}.$$

4.(a) The first thing to do when trying to fit a function to data is to plot the data points to see if you can spot any trends or patterns in the data that might suggest a particular type of equation. You can do this with a minimum of fuss on your graphing calculator. Entering the data for private university tuition into a TI-83 and producing a STATPLOT gives a graph like the one shown below.



This plot shows an increasing, concave up trend suggesting that either an exponential function or a power function with power $p > 1$ might do a reasonable job of representing this relationship.

On the basis of "common sense" I would think that an exponential function might be a slightly better choice than a power function. The reasoning process for this is: private universities have always charged tuition, so the function representing the relationship between tuition costs and years should never cross the x-axis. An exponential function never crosses the x-axis, whereas a power function (with power $p > 0$) goes through the point (0, 0).

On this basis, you could use EXPREG on a graphing calculator to find the equation for the exponential function that most closely matches the data points. (Note: instead of just entering years as "1971," "1998" etc., years have been entered as "71" or "98" here.) The output from performing EXPREG on a calculator is shown below.

```
ExpReg
 y=a*b^x
 a=6.968793398
 b=1.08201633
 r²=.9895405988
 r=.9947565525
```

On the basis of this, the equation relating private university tuition (T in dollars) to academic year (Y measured in years since 1900) would be:

$$T = 6.9688 \cdot (1.08201633)^{Y}.$$

Another criteria that you could have used is the value of the correlation coefficient, r, from the calculator. For an increasing set of data, the closer r is to 1, the closer the function matches the data. (For a set of data with a decreasing trend, the closer r is to -1, the closer the match between the function and the data.) In this approach, you would simply try all of the different kinds of regression (linear, exponential and power) that could conceivably be compatible with the trend that the STATPLOT shows and choose the one with the most favorable value of r. (See below.)

```
LinReg
 y=ax+b
 a=480.3390918
 b=-33707.81935
 r²=.9409404018
 r=.9700208255
```

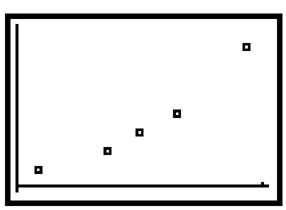
```
ExpReg
 y=a*b^x
 a=6.968793398
 b=1.08201633
 r²=.9895405988
 r=.9947565525
```

```
PwrReg
 y=a*x^b
 a=1.0093221E-9
 b=6.616144781
 r²=.9935566844
 r=.9967731359
```

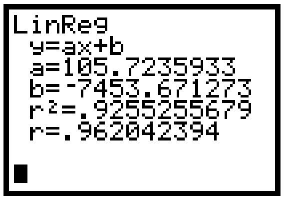
According to this, there is a slightly better match between the data and a power function. So, if you were using the correlation coefficient as your criteria, your equation would look like:

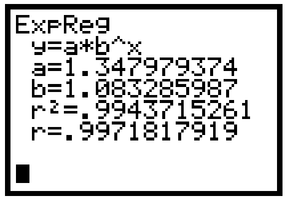
$$T = (1.0093 \times 10^{-9}) \cdot Y^{6.616}.$$

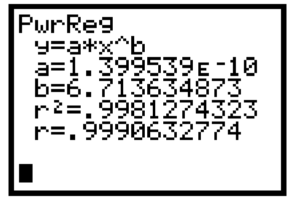
4.(b) Plotting the data for public university tuition gives on a calculator will give the graph shown in the STATPLOT below.



This plot shows an increasing, concave up trend again suggesting that either an exponential or a power function (with power $p > 1$) will do a reasonable job of representing the relationship between year and tuition. The results of performing linear, exponential and power regression are shown below.







Based on the values of the correlation coefficients, there is a slightly better match between a power function and the data than between an exponential function and the data (but not much). The power function that best represents the relationship between public university tuition (P in dollars) and years since 1900 (Y) would be:

$$P = (1.399\times10^{-10})\cdot Y^{6.714}.$$

4.(c) The idea of this problem is to figure out which years the students will be in college, use the functions that you found in Parts (a) and (b) to work out the tuition for each of those years, and then total them up.

To work out the results displayed in the table below, I used the power functions found in Parts (a) and (b). So long as you have used the functions that you found in Parts(a) and (b) in the manner described above, you will have done the problem correctly.

Student	Years in college	Year 1 tuition ($)	Year 2 tuition ($)	Year 3 tuition ($)	Year 4 tuition ($)	Total tuition ($)
A	2000, 2001, 2002, 2003	17219.49	18391.22	19629.94	20938.78	76179.43
B	2025, 2026, 2027, 2028	16767.62	17689.09	18653.31	19661.90	72771.93
C	2025, 2026, 2027, 2028	75366.25	79445.96	83711.61	88170.12	326693.94

5. In this question, the function is the dive computer's prediction of how long you have to wait between diving and flying.

- The independent variable (input to the function) is the number of minutes that you spend underwater.
- The dependent variable (output from the function) is the number of hours that the computer says you have to wait before flying.

Therefore, changes to the number of minutes that you spend underwater will be made on the *inside* of the function notation. Changes to the hours that the computer tells you to wait will be made on the *outside* of the function.

5(a). My wait time will be: $W(t + 15)$.

To avoid the notice of the tiger shark, I had to spend an extra 15 minutes under water. This means that the total amount of time spent underwater is the time spent diving, "t," plus the extra 15 minutes. So, the input to the computer will be $t + 15$.

5.(b) My wait time will be: $W(t) + 3$.

As my body needs longer to adjust, I need more hours of wait time before it is safe for me to fly. The computer says that after "t" minutes underwater, I need to wait $W(t)$ hours. I know that in addition to the time predicted by the computer, I need three additional hours, so my total wait time will be $W(t) + 3$.

5.(c) My wait time will be: $W(t - 5) - 2$.

In this situation there are two changes going on. Firstly, there is a change to my time underwater. This will be reflected by a change to the symbols that appear "inside" the function notation. I reduce my time underwater by five minutes. That means instead of spending "t" minutes underwater, I spend $t - 5$ minutes. This is how much time the dive computer records as being underwater, so its prediction for my wait time will be $W(t - 5)$ hours. The second change that is going on is that I am using a special blend of breathing gases instead of compressed air. These special gases mean that I won't have to wait as long as the computer predicts before I can fly. With the special gases, my actual wait time is two hours less than predicted by the computer, so my actual wait time will be $W(t - 5) - 2$.

5.(d) The time that I spend underwater will be the value of T in the equation: $W(T) = M$.

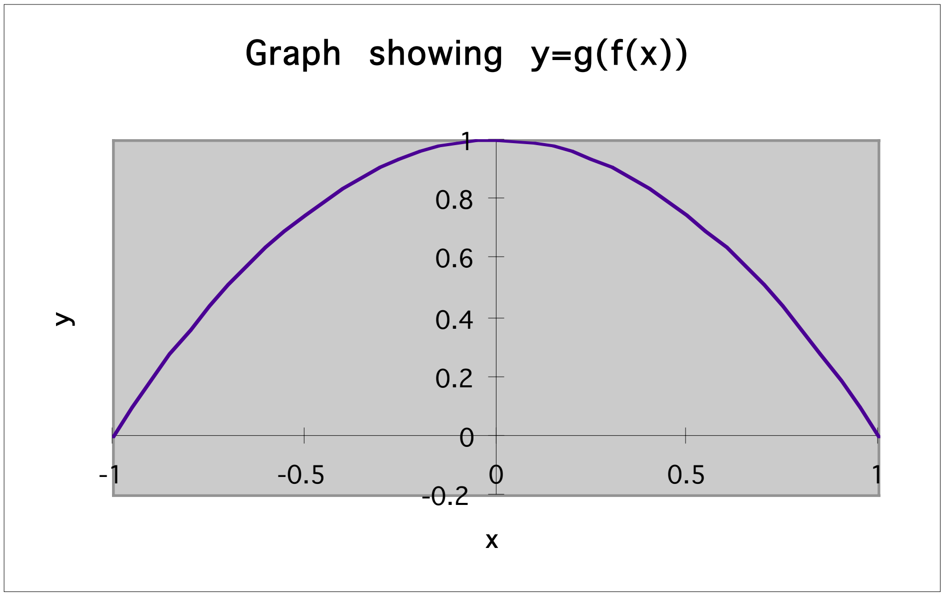
This question is different from Parts (a)-(c). Parts (a)-(c) ask you to find the time (in hours) that you have to wait between diving and flying. To answer this, you are supplied with information about the amount of time that you spend underwater (minutes) and asked to find a symbolic expression for the wait time. This question turns the situation around: it gives you the computer prediction of wait time (M hours) and asks you to express the time spent underwater. If you spend t minutes underwater then the computer will predict a wait time of $W(t)$ hours. You know that the predicted wait time is supposed to be M, so $W(t) = M$. The time that you are after is the value of t that makes this equation work.

6. First, we note the domains of the two functions f and g. These domains will affect the answers to Parts (a)-(d).

- Domain of f: All numbers between $x = -2$ and $x = 3$ including both end-points.

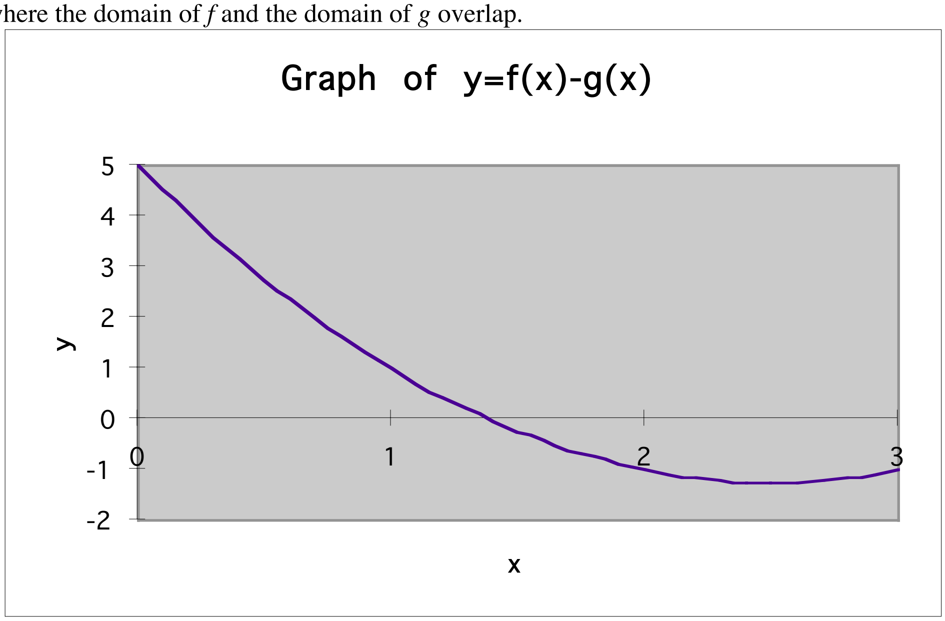
- Domain of g: All numbers between $x = 0$ and $x = 4$ including both end-points.

6.(a) The graph of $y = g(f(x))$ concentrating on the interval between $x = -1$ and $x = 1$ is shown below.



As an aside, the reason for concentrating on the interval between $x = -1$ and $x = 1$ is that when x is between these values, the output $f(x)$ is between zero and three. Since the domain of g is the interval [0, 4], concentrating on x between $x = -1$ and $x = 1$ will guarantee that the outputs produced by the function f will be suitable inputs for the function g.

6(b). The graph of $y = f(x) - g(x)$ is shown below. An important feature of the graph is that y-values are only present when x is between $x = 0$ and $x = 3$ (inclusive). This is because you can only find the difference of the functions f and g when both are defined. As a result, the difference $f - g$ is only defined on the interval where the domain of f and the domain of g overlap.



6.(c) The new functions h and p are defined by the equations:

$$h(x) = \frac{f(x)}{g(x)} \quad \text{and} \quad p(x) = \frac{g(x)}{f(x)}.$$

The new functions h and p will be defined only in the overlap of the domains of the functions f and g. That means that, at most, the new functions will be defined between $x = 0$ and $x = 3$. Therefore, all points between $x = -2$ and $x = 0$ (including $x = -2$ but not $x = 0$) are excluded from the domains of h and p. Likewise, all points between $x = 3$ and $x = 4$ (including $x = 4$ but not $x = 3$) will be excluded from the domains of h and p.

When forming the quotient of two functions, the new function will not be defined at points where the function on the bottom of the quotient (i.e. the denominator) is equal to zero.

Therefore, h will not be defined at $x = 1$ and $x = 3$, so these points are also excluded from the domain of the function h.

Likewise, the function p will not be defined at $x = 2$, so this point is also excluded from the domain of p.

6.(d) The graph of $y = h(x)$ is shown below. Note that the function h does not have any values except between $x = 0$ and $x = 3$, and also that the function h does not have a value at $x = 1$ or $x = 3$.

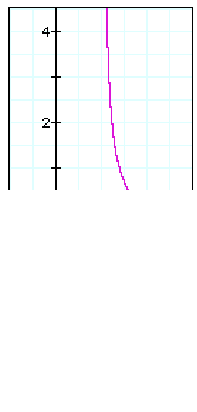
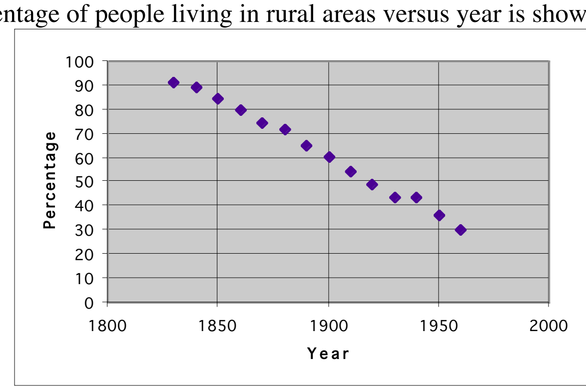


Figure 3: Graph of $y = h(x)$.

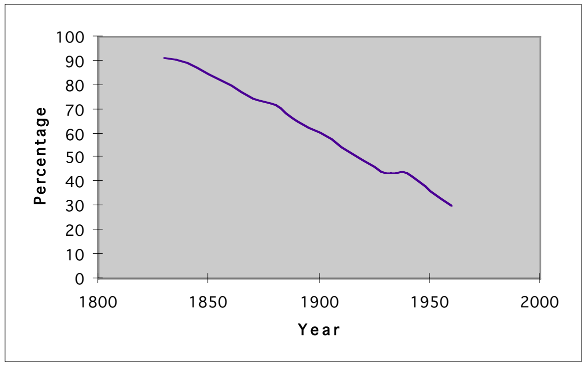
8.(a) The plot of percentage of people living in rural areas versus year is shown below.



A function that was going to represent this relationship would need, first of all, to be **decreasing.** This is because as you read the graph from left to right, you see that the height drops over each decade.

A function that was going to do a reasonable job of representing this relationship would not necessarily have to show a great deal of **concavity**. This is because all of the points appear to lie in a pattern that is quite close to a straight line.

If you were determined to find a function that fit the data points perfectly, then you could sketch a smooth curve through the points shown above (see the curve sketched below) and then inspect the smooth curve to see where it is concave up and concave down.



Inspection of this smooth curve suggested the intervals given in the table below.

Intervals over which the function should be concave down	Intervals over which the function should be concave up
(1830, 1870)	(1870, 1880)
(1880, 1890)	(1890, 1900)
(1900, 1910)	(1910, 1940)
(1940, 1950)	(1950, 1960)

8.(b) Because the points shown in Part (a) do not deviate that much from a straight line, a linear function will probably be a reasonable representation of this relationship. Calculating a linear function to do this:

Calculating the slope: $m = \dfrac{30.1 - 91.2}{1960 - 1830} = -0.47$.

Calculating the intercept: $y = m*x + b$

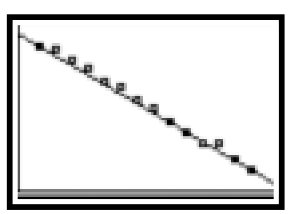
$91.2 = -0.47*1830 + b$

$b = 951.30$

Put the equation together: $y = -0.47*x + 951.30$

where y is the percentage and x is the year.

Plotting both the data and the function on the same set of axes gives the graph shown below.



This shows that the outputs from the function quite closely matches the values of the data points all the way from 1830 to 1960. To see how far outside that range you could possibly go, observe that as the output for the function is the percentage of the US population living in rural areas, it can be safely assumed that the outputs from the function should not rise above 100% and should not fall below 0%. Solving to find the years that the function attains these values:

$$100 = -0.47*x + 951.30, \quad \text{so that:} \quad x = 1811.28$$

$$0 = -0.47*x + 951.30, \quad \text{so that:} \quad x = 2024.04$$

The largest set of x-values for which the outputs of the function could be considered reasonable would therefore be: (1811.28, 2024.04).

8.(c) If you substitute $x = 2050$ into the equation from Question 4, then you obtain:

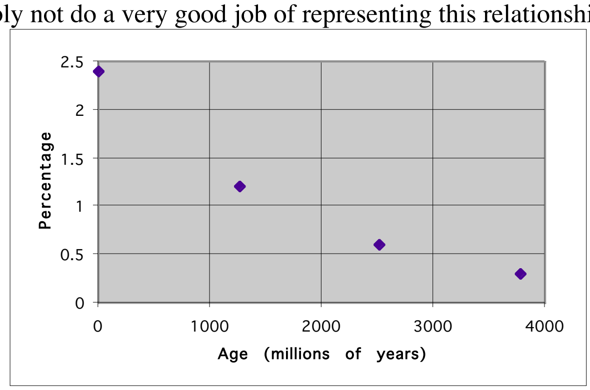
$$y = -0.47*2050 + 951.30 = -12.2.$$

This result does not make any sense because you can have a negative percentage of the US population. The reason for this (as indicated by the calculations in Part (b)) is that the "problem domain" of the linear function is, at best, the interval: (1811.28, 2024.04). Since $x = 2050$ lies well outside this problem domain, there's no real reason to think that the output from the function will resemble the actual percentage or people living in rural areas in the US in the year 2050.

9.(a) The description of the problem includes the information that fresh sediment will normally contain about 2.4% of potassium-40, and that the half-life of potassium-40 is 1260 million years. This means that each 1260 million years, the percentage will halve. From this information you can make a table relating the age of the rock to the percentage of potassium-40 (see table below).

Age (millions of years)	0	1260	2520	3780
Percentage of potassium-40	2.4	1.2	0.6	0.3

If you plot the data in this table, you get a graph resembling the one shown below. From the appearance of this graph, you can see that the points do not appear to line up and form a straight line, suggesting that a linear function will probably not do a very good job of representing this relationship.



Instead, this graph has a decreasing but concave up appearance that suggests that an exponential decay function might do a reasonable job of representing the relationship. To test this hypothesis, you calculate the growth factor B from the equation for an exponential function:

$$y = A \cdot B^x$$

using different pairs of points. If you always obtain the same growth factor, B, then an exponential function will do an excellent job of representing this relationship. Using the points (0, 2.4) and (1260, 1.2) to calculate B gives:

$$B = \left(\frac{1.2}{2.4}\right)^{\frac{1}{1260}} = 0.9994500345.$$

Using any other pair of points to calculate B also gives B = 0.9994500345. As you obtain the same value for the growth factor regardless of the points that are used to calculate it, an **exponential function** will do an excellent job of representing the relationship between the age of the rock and the percentage of potassium-40.

9.(b) As calculated, the growth factor, B = 0.9994500345. To find the initial value, A, you can substitute known values for x and y and the value of the growth factor into the equation for an exponential function:

$$y = A \cdot B^x.$$

Using the point (1260, 1.2) gives:

$$A = \frac{1.2}{(0.9994500345)^{1260}} = 2.4.$$

(Note that the value of A is simply the percentage when $x = 0$.) Therefore, the equation for the exponential function that represents the relationship between the age of the rock (x, measure in units of millions of years) and the percentage of potassium-40 (y) is:

$$y = 2.4 \cdot (0.9994500345)^x.$$

9.(c) One way to find the age of the *C. megalodon* tooth is to graph the exponential function on a graphing calculator, and then use INTERSECT or TRACE to find the point on the graph with y-coordinate of the graph is equal to 2.38683. This gives that the *C. megalodon* tooth is approximately 10 million years old.

You can also use logarithms to find the age of the tooth. The equation that you need to solve is:

$$2.38683 = 2.4 \cdot (0.9994500345)^x$$

to find the value of x. To start with, simplify as much as possible by dividing both sides by 2.4.

$$\frac{2.38683}{2.4} = (0.9994500345)^x.$$

Now take logarithms of both sides and apply the super fun happy rule:

$$\log\left(\frac{2.38683}{2.4}\right) = x \cdot \log(0.9994500345).$$

Finally, divide both sides of the equation by log(0.9994500345) and evaluate the logarithms on a calculator.

$$x = \frac{\log\left(\frac{2.38683}{2.4}\right)}{\log(0.9994500345)} = 10.00262448.$$

So, the megalodon tooth is about 10 million years old.

10.(a) The key phrase is the one that says the MnO_2 is deposited on the teeth at a constant rate. "Constant rate" is the defining property of a linear function.

10.(b) As mentioned in Part (a), there are two key facts to realize here.

- The *rates of change* that you are given here are constant. This means that you are looking for linear functions.

- When they are part of a living creature, teeth do not have any manganese dioxide (MnO_2) on them. The manganese dioxide deposits begin to accumulate when the tooth ceases to be part of a living organism (i.e. it is either lost or else the animal dies). This means that the linear functions that you are looking for should have intercepts of zero.

Let T denote the thickness of the MnO_2 layer in millimeters, and A be the age of the tooth in years. Then the two equations will be:

Rate of change = 0.15 mm per thousand years: $T = 0.00015 \cdot A$.

Rate of change = 1.4 mm per thousand years: $T = 0.00014 \cdot A$.

10.(c) The idea in this problem is to take the values of T used by Dr. Tschernezky and then used the equations found in Part (b) to solve for A.

Tooth 1: T = 1.7mm.

With a rate of change of 0.15 mm per thousand years:

$$A = \frac{1.7}{0.00015} = 11{,}333.33.$$

With a rate of change of 0.14 mm per thousand years:

$$A = \frac{1.7}{0.0014} = 1{,}214.2857.$$

So, according to this analysis the first tooth would be between approximately 1,200 and 11,000 years old.

Tooth 2: T = 3.7mm.

With a rate of change of 0.15 mm per thousand years:

$$A = \frac{3.7}{0.00015} = 24{,}666.67.$$

With a rate of change of 0.14 mm per thousand years:

$$A = \frac{3.7}{0.0014} = 2{,}642.857.$$

So, according to this analysis the second tooth would be between approximately 2,600 and 25,000 years old.