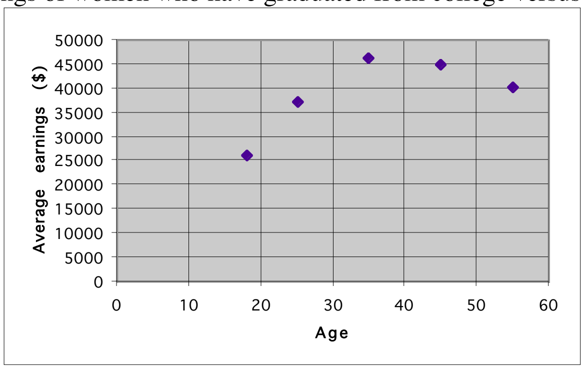
Solutions: Test #1, Set #2

These answers are provided to give you something to check your answers against and to give you some idea of how the problems were solved. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.

1.(a) The average earnings of women who have graduated from college versus age is shown below.



The simplest polynomial function that can exhibit a shape (one hump - a hill) like that shown in the plot is a quadratic function with an equation like:

$$y = a \cdot x^2 + b \cdot x + c.$$

1.(b) Entering the data into a calculator and performing a quadratic regression gives the equation shown below.

$$E = -37.47 * A^2 + 3089.14 * A - 16817.16$$

In this equation, *A* represents the age (in years) and *E* is the average earnings.

1.(c) To put the equation obtained in Part (b) into **vertex form**, you perform the process called "Completing the Square." The work for this is shown below.

$$E = -37.47*(A^2 - 82.443*A + 448.816)$$

$$E = -37.47*(A^2 - 82.443*A + 1699.212 - 1699.212 + 448.816)$$

$$E = -37.47*((A - 41.2215)^2 - 1699.212 + 448.816)$$

$$E = -37.47*((A - 41.2215)^2 - 1250.396)$$

$$E = -37.47*(A - 41.2215)^2 + 46852.39.$$

1.(d) From the vertex form, the coordinates of the vertex of the graph of the quadratic function are approximately (41.22, 46852.39). As these are the coordinates of the vertex, and the graph of the function is an upside-down parabola, they are also the coordinates of the highest point on the graph. Therefore:

- The average earnings of women who are college graduates attains a peak at an age of approximately 41.22

• The maximum average earnings of women who are college educated is approximately $46852.39 per year.

2.(a) $y = \frac{1}{18}(x+3)^2(x-1)(x-2)$

2.(b) $y = \frac{1}{2}(x-1)^3(x+1)(x+2)$

2.(c) $y = \frac{1}{4}(x+2)(x-1)(x-2)$

2.(d) $y = \frac{1}{48}(x-3)(x-1)^2(x+2)^3$

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3. **Please note:** There are many possible solutions to problem 3. The solutions presented here are not the only ones that are possible. The main things that we are looking for in your solution include:

• Some evidence of a rational thought process while working through the problems

• A thought process that most people would regard as a reasonable way to answer the questions on the homework assignment, and,

• Enough work and sufficiently clear explanations so that a reasonably well-educated person could make sense out of what you have done.

If you handle this problem with level-headedness and calm determination now, then you will find yourself extremely well-prepared when you have to do this sort of thing for real on the actual midterm.

3.(a) The completed version of Table 1 is shown below. Of all the polynomial functions that we are familiar with so far in Math Xa, a quartic function probably does the best job of representing share price as a function of time. This is because a quartic is the only polynomial function that we have seen that seems to be capable of having a "hump" followed by a "plateau" after it. (Remember the appearance of your graphs from Part 1 of the Biomedical Data lab - they looked a little like the graph of share price versus time.) This possibility is shown as the purple curve in Figure 1(a) below. (Two other possibilities are shown in Figures 1(b) and 1(c).) Note that none of the curves precisely matches the graph of share price, but that each goes some way towards capturing some aspect of the "large-scale pattern" or "general trend" of the share price graph.

Date	AOL Time Warner Share Price ($)
12/1997	6
12/1998	24
12/1999	84
12/2000	42

Completed Version of Table 1.

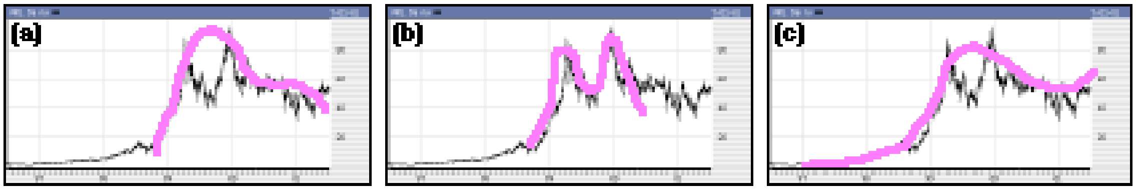


Figure 1: Three ways of imagining a quartic function whose graph might capture some aspects of the shape and behavior of the AOL-Timw Warner share price graph.

To find an equation for a quartic polynomial on a calculator, at least five data points are needed. In addition to the data points from Table 1, we used the point for July, 2000, when the share price was $62. Here are the meanings of the variables that we used:

- P = dependent variable = share price of AOL-Time Warner in dollars.
- t = independent variable = number of years since December 1998.

The values that we entered into a calculator were:

t	−1	0	1	1.5	2
P	6	24	84	62	42

to obtain the equation:

$$Y = 24.267 \cdot t^4 - 72.533 \cdot t^3 - 2.367 \cdot t^2 + 111.533 \cdot t + 24.$$

3.(b) A plot of EPS versus time is shown in Figure 2 below.

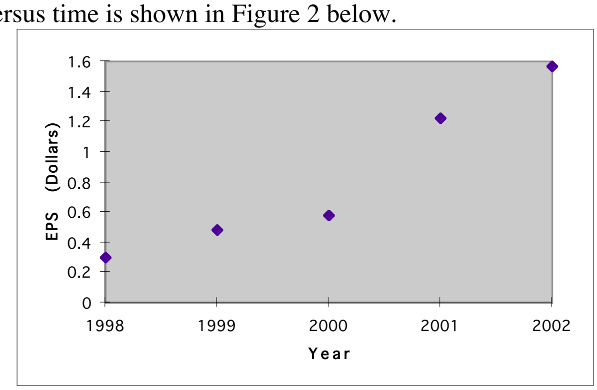


Figure 2: EPS as a function of time for AOL-Time Warner.

Based on the distribution of the points in Figure 1, it is possible to make a reasonable case that either a linear function or a quartic polynomial could do a reasonable job of representing the relationship between EPS and year. (In fact, you could probably make a pretty good argument that any polynomial would be okay here; the plots suggesting linear or quartic are shown in Figure 3 below.)

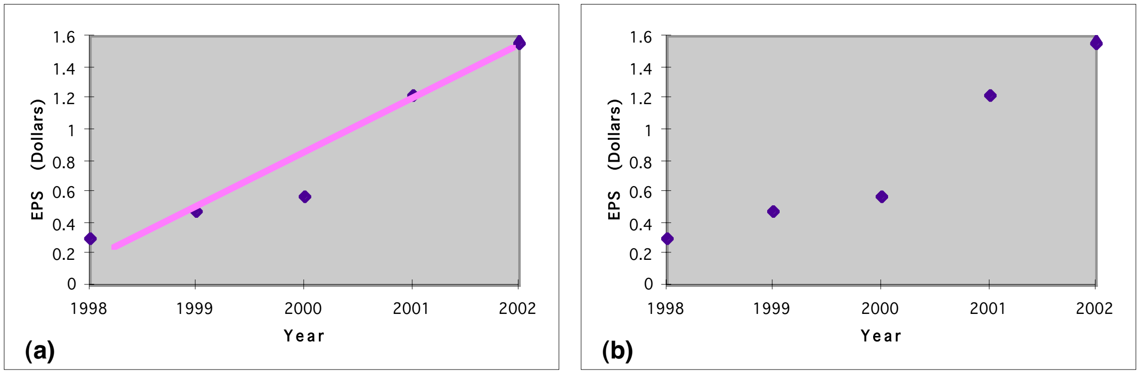


Figure 3: Using a linear function (a) or a quartic polynomial function (b) to represent the relationship between EPS and year.

When finding an equation, the variables that we used were:

- E = dependent variable = EPS in dollars.
- t = independent variable = number of years since December, 1998.

The values that we entered into a calculator to find the equation are shown in the Table below.

t	0	1	2	3	4
E	0.3	0.48	0.58	1.22	1.57

The equations that we obtained were:

Linear: $E = 0.328 \cdot t + 0.174$.
Quartic: $E = -0.0604 \cdot t^4 + 0.466 \cdot t^3 - 1.015 \cdot t^2 + 0.789 \cdot t + 0.3$

3.(c) There are (at least) two ways to find an equation for the PE ratio as a function of time. You could use the data provided in the graph of share price and the table of EPS values to calculate some values for the PE ratio and then try to fit a polynomial equation to this. A second way to work out an equation for PE ratio is to divide the equation for share price (from Part (a)) by the equation for EPS (from Part (b)). The critical thing that must happen for you to be able to do this is that you have to use the same time variable in both Parts (a) and (b). If you used a linear function for the EPS, you would obtain:

$$PE = \frac{24.267 * t^4 - 72.533 * t^3 - 2.367 * t^2 + 111.533 * t + 24}{0.328 * t + 0.174}.$$

If, instead, you were using a quartic polynomial for the EPS, you would obtain:

$$PE = \frac{24.267 * t^4 - 72.533 * t^3 - 2.367 * t^2 + 111.533 * t + 24}{-0.0604 * t^4 + 0.466 * t^3 - 1.015 * t^2 + 0.789 * t + 0.3}.$$

When AOL-Time Warner makes no profit, their EPS will be zero. When the denominator of a fractional function is zero, the graph of the function will show a vertical asymptote. So, when AOL-Time Warner does not make a profit, the graph of PE ratio versus time should show a vertical asymptote.

4.(a) The domains and ranges of the two functions are given below.

• Function $f(x)$: The **domain** consists of all numbers greater than or equal to zero. The **range** consists of all numbers greater than or equal to zero.

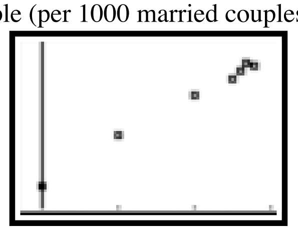
• Function $g(x)$: The **domain** consists of all numbers. The **range** consists of all numbers greater than or equal to zero.

4.(b) The equation for the new function is: $f(g(x)) = |x|$. The **domain** of the new function consists of all real numbers. The **range** of the new function consists of all numbers greater than or equal to zero.

4.(c) The equation for the new function is: $g(f(x)) = x$. The **domain** of the new function consists of all numbers that are greater than or equal to zero. The **range** of the new function consists of all numbers that are greater than or equal to zero.

4.(d) Despite first impressions, the two new functions are not precisely the same. $f(g(x))$ is defined even when x is less than zero; $g(f(x))$ is not even defined when x is less than zero.

5.(a) A plot of number of divorced people (per 1000 married couples) versus year is shown below.



5.(b) Based on the appearance of the plot in part (a), a linear function should do a very good job of representing the trend in the data. This is because the data points lie very close to a straight line.

5.(c) Using T = years since 1970 as the independent variable and D = number of divorced people (per 1000 married couples), linear regression on a calculator gives: $D = 4.555 \cdot T + 49.92$.

5.(d) The two parameters are the slope $m = 4.555$, and the intercept, $b = 49.92$. The interpretation of the intercept is that this is how many divorced people (per 1000 married couples) there were in 1970. The interpretation of the slope is that every year since 1970, there have been about 4.555 more divorced people (per 1000 married couples).

5.(e) Since the dependent variable is the number of divorced people per 1000 married couples, the highest value that this could have would be positive infinity. The conditions that would bring this into being would be everyone who is now married getting divorced, and no-one else getting married. Although this is theoretically possible, I doubt that this situation would arise in the foreseeable future. (Although this is a scenario envisioned by many science fiction writers for the distant future.)

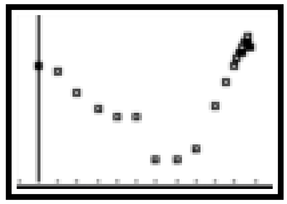
6.(a) Using A = size of the farm in acres as the dependent variable: $A = 139 \cdot (1.01)^T$.

6.(b) Substituting $T = 86$ into the equation gives: $A = 327.08$ acres.

6.(c) It will take about 70 years for the farm to double in size.

6.(d) It will take approximately 1653 years for the farm to grow to the point where it covers the entire continental United States.

7.(a) A plot of the median age of males at the time of their first marriage is shown below.

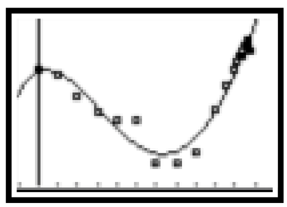


7.(b) Based on the appearance of the plot, I would suspect that a cubic function might do a good job of representing this data. This is because there is one obvious "valley" in the data. In addition, the data points near 1890 resemble half of a "hill." You could also say the around 1930-1940, it looks as though there is another "valley." In this case, a quartic function might be the best to use here. I will use a quartic function to represent this data.

7.(c) If T = years since 1890 is used as the independent variable and A = median age at time of first marriage is used as the dependent variable, then quartic regression on a calculator gives:

$$A = -0.00000016\cdot T^4 + 0.00005\cdot T^3 - 0.0037\cdot T^2 + 0.0299\cdot T + 25.969$$

7.(d) Plotting both the data points and the quartic equation gives a graph like the one shown below.



Between 1890 and 1930, the graph of the polynomial is very close to the data points. Likewise, between 1980 and 1998, the graph of the polynomial is very close to the data points. Between 1930 and 1980, the graph of the polynomial function is not far from the data points, but it is not as close as during the other time periods.

7.(e) Substituting $T = 111$ into the equation of the quartic gives: $A = 27.81$ years of age.

7.(f) Tracing the graph of the polynomial function gives that the minimum occurs at $T = 63.28$, and that the minimum value of A is 23.12 years. So, the minimum median age during the 20th century is 23.12 years, and this occurs during 1953. This minimum value does match the data in terms of roughly what year the minimum should occur in, but because this is a time interval when the polynomial graph does not represent the data very accurately, the value predicted for the median age is higher than the lowest value given by the data.

8.(a) $x = -4$, 0 and 1.

8.(b) $k = 1$.

8.(c) This expression cannot be evaluated. The reason is that the inverse of the function f is not a function in its own right. (This is mainly due to the horizontal line segment between $x = 0$ and $x = 2$ which fails the horizontal line test.) Therefore, there is no unambiguous value that can be assigned to the symbols: $f^{-1}(1.5)$.

9.(a) Using T = age of sample in years as the independent variable, and M = mass of carbon-14 in micrograms (μg) as the independent variable, an equation would be:

$$M = (0.0001) \cdot (0.9998790392)^T.$$

9.(b) Without doing further calculations, it is very difficult to say. On one hand, the amounts of carbon-14 remaining in the three mummies are quite similar, which might suggest that they all died at about the same time. On the other hand, in an exponential decay function (such as the one defined in part (a)), very small changes in the amount of carbon-14 present can take a very long time to occur. Although exponential decay functions usually decay rapidly at first, they soon slow down, and the decay happens less quickly.

9.(c) The ages of the mummies are given in the table below.

Mummy	Approximate age (years)
The Baby	3510
Cherchen Man	3008
The Beauty of Loulan	4005

9.(d) Given that Marco Polo wasn't even born until 1254 CE[1] (i.e. about 750 years ago), and the Silk Road was established after Marco Polo visited China, then the Silk Road must be less than 750 years old. If these people had died while travelling along the Silk Road, then they would have to be less than 750 years old. The carbon-14 dating suggests that they are all at least 3000 years old, which seems to make the theory that they were travelers on the Silk Road implausible.

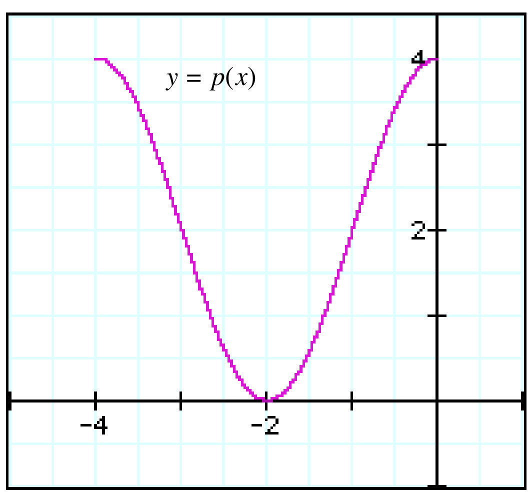
10.(a) The function $h(x)$ is increasing on the interval (2, 4). The function $h(x)$ is decreasing on the interval (0, 2).

10.(b) The function $h(x)$ is concave up on the interval (1, 3). The function $h(x)$ is concave down on the intervals (0, 1) and (3, 4).

10.(c) The zeros of $h(x)$ are located at $x = 1$ and $x = 3$.

10.(d) The domain of the new function $p(x)$ is the interval $[-4, 0]$. The range of the new function $p(x)$ is the interval $[0, 4]$.

10.(e) The graph of $y = p(x)$ is shown in the diagram below. The only zero of the function $p(x)$ occurs at $x = -2$.



[1] That is, 1254 of the Common Era.