Practice Problems: Test #1, Set #3

Important Information:

1. The first test will be held on **Wednesday October 30 from 7-9pm in Science Center D**.

2. The test will include approximately eight problems (each with multiple parts).

3. You will have 2 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test.

5. The specific topics that will be tested are:

   - The definition of a function.
   - Representations of functions (graphs, numbers/tables, written descriptions, equations).
   - Interpreting graphs, tables, words and symbols.
   - Modeling relationships using linear functions.
   - Interpreting the parameters of a linear function.
   - Calculating and interpreting rates of change.
   - Exponential growth and exponential functions.
   - Modeling relationships using exponential functions.
   - Solving equations involving linear and exponential functions (logarithms).
   - Rates of change and concavity.
   - Approximating functions using their rates of change (Euler’s method).
   - Transformations of functions (shifts, stretches and reflections of functions).
   - Power functions.
   - Predicting the appearance of the graph of a power function.
   - Modeling relationships using power functions.
   - Polynomial functions.
   - Predicting the appearance of the graph of a polynomial, exponential or power function.
   - Fitting a polynomial function to data.
   - Compositions of functions.
   - The concept of the inverse of a function.
   - Functions defined in pieces.

6. The problems included here have been chosen because they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

7. Good places to go for help include:

   - Office hours.
   - The labs on Tuesday 10/29.
   - The Math Question Center
   - The course-wide review on Tuesday 10/29 from 5:00-7:00pm in Science Center D.

8. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

1. Cardiovascular diseases (diseases of the heart and blood vessels) are America's number one killer[1]. Almost 25% of the population – more than 60 million people - suffer from some form of cardiovascular disease. During 1996, deaths due to heart disease represented 41.4% of all deaths, with one death occurring (on average) every 33 seconds for the entire year[2].

Death rates due to heart disease vary according to age, gender, race and geographical location. Generally speaking, men have a higher death rate from heart disease than do women. Minorities often suffer from a higher death rate due to heart disease than does the white population. Geographically, the state with the lowest death rate due to heart disease is Utah, followed by Colorado, Hawaii, New Mexico and Alaska. Mississippi has the highest death rate due to heart disease, followed by South Carolina, Tennessee, the District of Columbia and Louisiana.

Perhaps the only good news regarding heart disease is that the death rate (per 100,000 people) has dropped steadily for the last 50 years. In 1960, for example, the national death rate was 369 per 100,000 people[3]. Studies conducted in 1995 suggested that the death rate had dropped to approximately 281 deaths (per 100,000 people). Two important factors that are thought to have contributed to the reduction of the cardiovascular death rate are the reduction in the number of people smoking, and a general reduction in the fat content of the average person's diet.

In this problem, T will represent the number of years since 1960, and $C(T)$ will represent the death rate due to heart disease at time T. The units of $N(T)$ are number of deaths for every 100,000 people living in America per year. The function $C(T)$ obeys the differential equation:

$$\frac{\Delta C}{\Delta T} = -0.02989 \cdot \left[C(T) - 250\right]$$

quite closely.

(a) In the introduction to this problem, it was noted that in 1960, the death rate due to heart disease was 369. Express this information using the mathematical symbols defined above.

(b) Use Euler's method and the table provided below to approximate the death rate due to heart disease in the year 2010.

Year	T	C(T)	$\frac{\Delta C}{\Delta T}$	$\frac{\Delta C}{\Delta T} \cdot \Delta T$	New C(T)
1960	0				
1970	10				
1980	20				
1990	30				
2000	40				

[1] Source: U.S. Department of Health and Human Resources.

[2] Source: U.S. Department of Health and Human Resources. *Health in the United States*. Washington, DC: Department of Health and Human Resources, 1998.

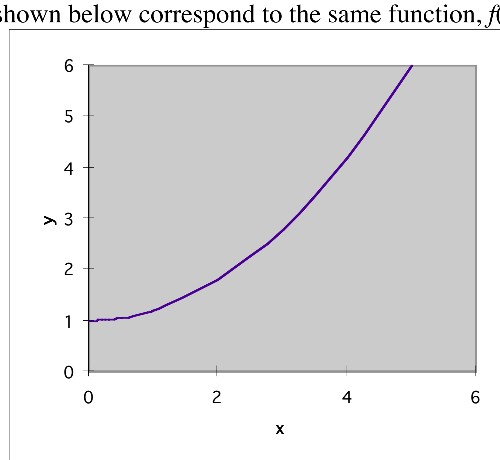
[3] Source: U.S. Department of Health and Human Resources.

(c) Do you think that $C(T)$ is an increasing or a decreasing function of time? How can you use the equation for $\frac{\Delta C}{\Delta T}$ to decide if $C(T)$ is an increasing or a decreasing function?

(d) Do you think that $C(T)$ is a concave up or concave down function? How can you use the entries in the table to check?

(e) If you were to sketch the graph of $C(T)$, would you expect the graph to have a horizontal asymptote? At what height would the asymptote be located?

2. The graph and table shown below correspond to the same function, $f(x)$.



x	0	1	5
f(x)	1	1.2	6

(a) Functions with graphs like the one shown above are often exponential functions: $y = A \cdot B^x$. Use the data in the table to decide whether $f(x)$ is an exponential function or not.

(b) Other functions with increasing, concave up graphs are power functions: $y = k \cdot x^p$ where the power p is greater than 1. Use the data in the table to decide whether $f(x)$ is a power function or not.

(c) One last possiblilty is that $f(x)$ could be an exponential function or a power function that has been vertically shifted up by 1 unit:

$$f(x) = 1 + k \cdot x^p \qquad \text{OR} \qquad f(x) = 1 + A \cdot B^x$$

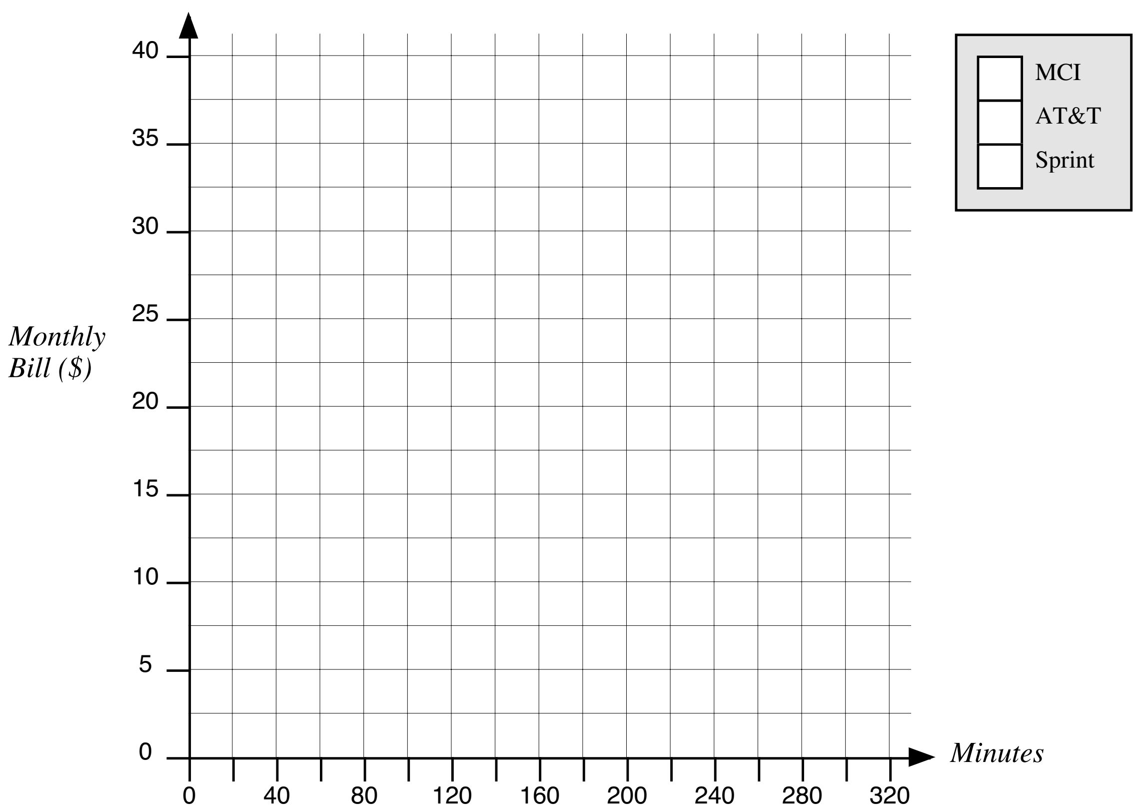
Find an equation for $f(x)$ that exactly matches the data given in the table. As part of your answer, you should show that your equation really does exactly match the values given in the table.

3. The table shown below gives the fixed monthly fee (in dollars) and the cost for each minute of three major long-distance calling plans[4].

Company	Plan	Fixed monthly fee ($)	Cost for each minute (cents)
MCI	12 cent Anytime	0	12
AT&T	OneRate	3.95	7
Sprint	Nickel Anytime	8.95	5

(a) Let T = number of minutes you spend calling long distance be the independent variable. Find equations to represent the total phone bill (fixed monthly fee plus cost of minutes) in dollars for each of the three long-distance calling plans.

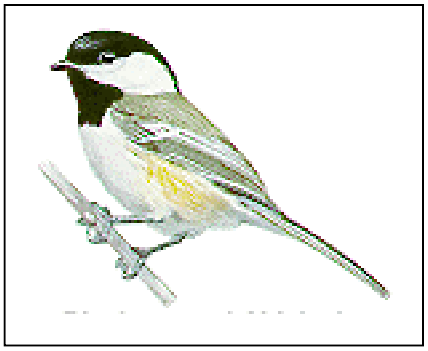
(b) Plot a graph showing total phone bill versus the number of minutes for each of the three plans. (Use the axes provided below.)



(c) MCI has the highest cost for each minute. Under what circumstances would the MCI plan be your most economical option?

(d) The Sprint plan has the lowest cost for each minute. How many minutes would you have to talk each month in order to make the Sprint plan the most economical?

[4] Source: http://www.myrateplan.com/

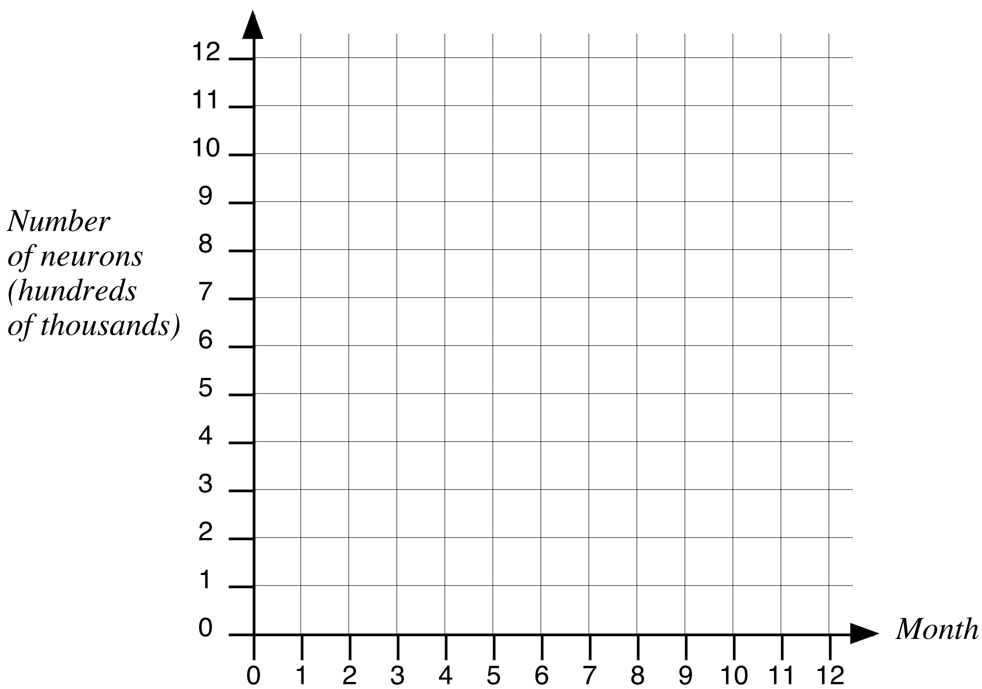


4. The Black-capped chickadee[5] (*Poecile atricapillus*) is a small bird that stores seeds (and later finds them again) for food. The table below gives the number of neurons in an average adult Black-capped chickadee hippocampus at different times of the year[6].

(a) Use the axes given below to plot the number of neurons versus the month.

(b) Based on the appearance of your graph, what kind of function would do a good job of representing the relationship between number of neurons and month?

Month	4	6	8	10	12
Number of neurons (hundreds of thousands)	9.2	7.5	9.4	11.4	8.7



(c) Find an equation for number of neurons as a function of month.

(d) Predict the number of neurons that an average adult Black-capped chickadee will have in it's hippocampus in January (i.e. month 1).

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[5] Picture by Larry McQueen. Image source: Cornell Laboratory of Ornithology, http://birds.cornell.edu/

[6] Source: Smulders, T.V., Shiflett, M.W., Sperling, A.J. and DeVoogd, T.J. (2000) Seasonal changes in neuron numbers in the hippocampal formation of a food-hoarding bird: The Black-capped chickadee." *Journal of Neurobiology*, **44**(4): 414-422.

5. Organic matter contains a radioactive isotope of carbon, known as carbon-14. The half life of carbon-14 is 5730 years. A 100g sample of fresh organic matter will normally contain 0.0001 μg of carbon-14. In this problem, you will examine some of the ways in which the smallest mistakes can drastically alter results when exponential functions and logarithms are involved.

(a) Find an equation to describe the amount of carbon-14 (in micrograms) that remains in a 100g sample of organic matter after T years.

(b) Suppose that a 100g sample of organic matter contains 0.0000327 μg of carbon-14. How old is the organic matter?

(c) Now, suppose that you misread the number from Part (b) and used the figure of 0.0000372 μg of carbon-14 instead. How far off would the age of the organic matter be?

(d) Lastly, suppose that your little buddy Barry calculated the age of a sample of organic matter and determined that it was 2000 years old. Looking over Barry's work, you notice that he incorrectly remembered the half life of carbon-14 to be 5370 years and used this figure in his calculations. What is the correct age of the organic matter?

6. Carbon-14 dating is an important method for establishing the age of artifacts and fossils in the fields of archaeology and paleontology. Technicians at a laboratory measure the amount, c, of carbon-14 present in an artifact, and obtain the age, A, by plugging c into the function $A = f(c)$. Match the stories with the algebraic expressions below.

(a) An ancient shroud was partially burnt, doubling the amount of carbon-14 in it.

(b) A recently hired employee did their sums wrong and got twice the actual age of an urn.

(c) The janitor used an antique bowl for an ashtray, and did not clean it properly, increasing the amount of carbon-14 present.

(d) The carbon-14 measuring machine was poorly calibrated, and its readings were 25% off.

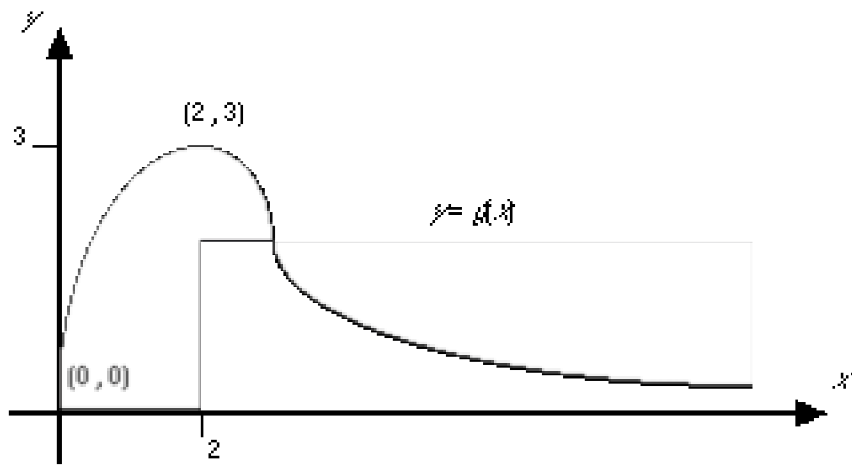
(e) A professor does not accept the laboratory's results, believing a relic to be far older. He publishes his opinion as the age of the relic.

(I) $f(c) + 1000$ (II) $2*f(c)$

(III) $f(0.75*c)$ (IV) $f(2c)$

(V) $f(c + 1)$

7. A function g is defined by the graph shown on the next page. Sketch an accurate graph of $y = g\left(\frac{-x}{3}\right)$ labeling all important points.



8. The table below gives the number of people (in thousands) receiving medicaid, and the payments made to medicaid vendors (in millions of dollars) between 1975 and 1997[7].

Year	Number of recipients (thousands)	Vendor payments (millions of dollars)
1975	3615	4358
1981	3367	9926
1985	3061	14096
1990	3202	21508
1995	4119	36527
1996	4285	36947
1997	3954	37721

(a) Plot a graph showing the average expenditure of medicaid per recipient of medicaid between 1975 and 1997. What kind of function would do a good job of representing the average expenditure on medicaid per recipient as a function of time?

(b) Find an equation for average expenditure of medicaid as a function of time.

(c) Plot a graph showing the number of recipients of medicaid versus year for 1975 and 1997. What kind of function would do a good job of representing the number of recipients of medicaid as a function of time?

(d) Find an equation for number of recipients of medicaid as a function of time.

(e) How could you combine the equations that you found in parts (b) and (d) of this problem to create an equation that would give the total expenditure on medicaid as a function of time?

(f) Plot a graph showing the expenditure on medicaid (in millions of dollars) versus the number of recipients of medicaid (in thousands). Based on the appearance of your plot, what kind of function would do a good job of giving the expenditure on medicaid as a function of the number of recipients of medicaid?

(g) Find an equation for medicaid expenditure as a function of the number of recipients of medicaid.

(h) How could you combine the functions that you found in parts (d) and (g) of this problem to create an equation for the total expenditure on medicaid as a function of time?

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[7] Source: Health Care Financing Administration, *2082 Report* (1999).

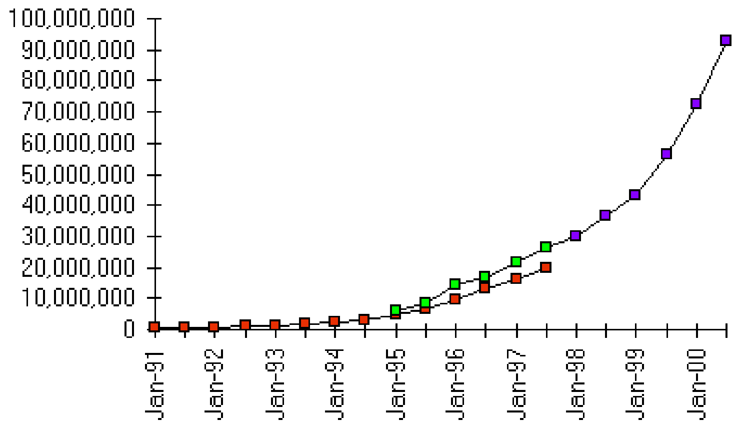
9. The following graphs (see next page) show the number of computers (or "hosts") connected to the internet between January 1991 and January 2000[8].

(a) Find the coordinates of two points on the graph. It is very difficult to tell whether the point (0, 0) is on the graph or not, so don't use this as one of your points.

(b) Based on the appearance of the graph, a power function or an exponential function might do a good job of representing the trends in the data. Find the equation of an exponential function and the equation of a power function, based on the two points that you found in part (a).

(c) If you use your exponential function to make the prediction, when will there be one billion (i.e. 1,000,000,000) computers connected to the Internet?

(d) If you use your power function to make the prediction, when will there be one billion (i.e. 1,000,000,000) computers connected to the Internet?



(e) Researchers at the National Institute of Supercomputer Applications at the University of Illinois at Urbana Champaign predicted that during the 1990's, the number of computers connected to the internet would increase by 10% per month[9]. What kind of growth (linear, exponential or power) does this prediction imply?

(f) Does the data shown in the graph support the predictions made by the researchers regarding the growth of the Internet? Briefly explain.

10. The table shown below gives the number of people living in the US who are aged 65 or older[10] from 1900-1990, and projections of the number of people aged 65 and older for 2000-2050.

Year	Number of people aged 65 or older (in thousands)
1900	3099
1910	3986
1920	4929
1930	6705
1940	9031
1950	12397
1960	16675

[8] Source: Internet Software Consortium, http://www.isc.org

[9] Source: Kanfer, A. (1998) "Easy to use software and the growth of the internet." Presentation available on-line from: http://www.ncsa.uiuc.edu/edu/trg/

[10] Source: US Bureau of the Census, *Population Projections of the U.S. by Age, Sex, Race, and Hispanic Origin*, 1996.

1970	20107
1980	25549
1990	31235
2000	34709
2010	39408
2020	53220
2030	69379
2040	75233
2050	78859

The data in this table is quite will represented by an exponential equation:

$$N = 3876 \cdot (1.02068)^T,$$

where N is the number of thousands of people aged 65 or older, and T is the number of years since 1900.

(a) Plot a graph showing the data points and the graph of the equation: $N = 3876 \cdot (1.02068)^T$.

(b) If you were to use the equation to make calculations about the size of the over 65 population, over what ranges of years do you think the results of the calculations would correspond most closely to reality?

(c) In what year did the over 65 population reach 20,000,000?

(d) The population of United States is approximately 285,000,000. According to the equation, in what year will the over 65 population reach 285,000,000?

(e) How much confidence do you have in the prediction from part (d)? Explain.

(f) If you carefully examine your plot from part (a), you will notice that the data points seem to be leveling off at the end of the twentieth century. However, immediately after the year 2000, the size of the over 65 population appears to shoot up. What do you think might be responsible for this?