

## Solutions: Test #1, Set #3

These answers are provided to give you something to check your answers against and to give you some idea of how the problems were solved. Remember that on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.

**1.(a)** The information that you have to represent here is the fact that in 1960, the death rate due to heart disease was 369 people per year (for every 100,000 people living in the United States). That is, when  $t = 0$ , the function is equal to 369. In symbols this means:  $C(0) = 369$ .

**1.(b)** The completed table is shown below.

Year	T	C(T)	$\frac{\Delta C}{\Delta T}$	$\frac{\Delta C}{\Delta T} \cdot \Delta T$	New C(T)
1960	0	369	-3.55691	-35.5691	333.4309
1970	10	333.4309	-2.4937496	-24.937496	308.493404
1980	20	308.493404	-1.7483678	-17.48367845	291.0097255
1990	30	291.0097255	-1.225780697	-12.25780697	278.7519185
2000	40	278.7519185	-0.859394845	-8.59394845	270.1579701

Therefore, the estimate for the death rate due to heart disease in 2010 is about 270.16 deaths per year (for every 100,000 people living in the United States).

**1.(c)** From the table in Part (b) it appears that  $C(T)$  is a decreasing function of time. This is because the rate of change is always negative, and when the rate of change is negative the function is decreasing. You can also see this from the structure of the equation for the rate of change,

$$\frac{\Delta C}{\Delta T} = -0.02989 \cdot [C(T) - 250].$$

When  $C(T)$  is greater than 250, the quantity  $[C(T) - 250]$  is positive. Therefore, when you are using the above equation to determine the rate of change, you will be multiplying a negative (i.e.  $-0.02989$ ) by a positive (i.e.  $[C(T) - 250]$ ) which will make the rate of change negative over all.

**1.(d)** If you have a look at how the numbers in the  $\frac{\Delta C}{\Delta T}$  column of the table change as you read down the column, you can see that they get larger as you go down. Therefore, the rate of change of the function is increasing so that the original function will be concave up.

**1.(e)** The paw-print of a horizontal asymptote is a rate of change that falls to zero. From the structure of the equation given for  $\frac{\Delta C}{\Delta T}$ , this will occur when  $C(T) = 250$ . Therefore, you would expect the graph of  $C(T)$  to show a horizontal asymptote, and the asymptote should be at a height of 250.

**2.(a)** To test to see whether the graph could be the graph of an exponential function you calculate the growth factor for different pairs of points and see whether you get the same result.

For (0, 1) and (1, 1.2) you get a growth factor of 1.2.

For (0, 1) and (5, 6) you get a growth factor of  $6^{1/5} = 1.430969081$ .

As the two growth factors are different, the formulas that you would get for the exponential function will be different in each case, and so  $f(x)$  cannot be an exponential function.

**2.(b)** The function cannot be a power function as the graph of a power function passes through the point (0, 0) provided the power is greater than zero. Since the graph of  $y = f(x)$  does not pass through (0, 0),  $f(x)$  cannot be a power function.

**2.(c)** It is not possible for  $f(x) = 1 + A \cdot B^x$  and for the graph of  $y = f(x)$  to pass through the point (0, 1). This is because if you plug  $x = 0$  and  $y = 1$  into such an equation then you get that  $A = 0$ . If  $A = 0$  then the equation becomes  $f(x) = 1$ . This is the equation of a horizontal line, not an increasing concave up function.

So, the equation of the function must be of the form:  $f(x) = 1 + k \cdot x^p$ . To determine the values of  $k$  and  $p$ , plug in some of the points from the table and solve the resulting equations. Using the points (1, 1.2) and (5, 6):

$$\begin{array}{rclcl} 1.2 & = & 1 + k \cdot 1^p & \text{so} & k = 0.2 \\ 6 & = & 1 + 0.2 \cdot 5^p & \text{so} & p = 2. \end{array}$$

Therefore, the equation for  $f(x)$  is:  $f(x) = 1 + 0.2 \cdot x^2$ .

**3.(a)** Let  $T$  = number of minutes used for long distance telephone conversations each month, and  $B$  = amount of the monthly long-distance bill in dollars. Then the three plans could be represented by the equations:

$$\text{MCI: } B = 0.12 \cdot T$$

$$\text{AT\&T: } B = 3.95 + 0.07 \cdot T$$

$$\text{Sprint: } B = 8.95 + 0.05 \cdot T.$$

**3.(b)** A plot showing the graphs for the three calling plans is shown on the next page.

**3.(c)** From the appearance of the plot in Part (b), the MCI plan is the most economical from  $T = 0$  to the intersection of the MCI graph with the AT&T graph. To locate this point, set the two equations equal to each other and solve to find  $T$ .

$$\begin{array}{rclcl} 0.12 \cdot T & = & 3.95 + 0.07 \cdot T \\ 0.05 \cdot T & = & 3.95 \\ T & = & 3.95/0.05 \\ T & = & 79. \end{array}$$

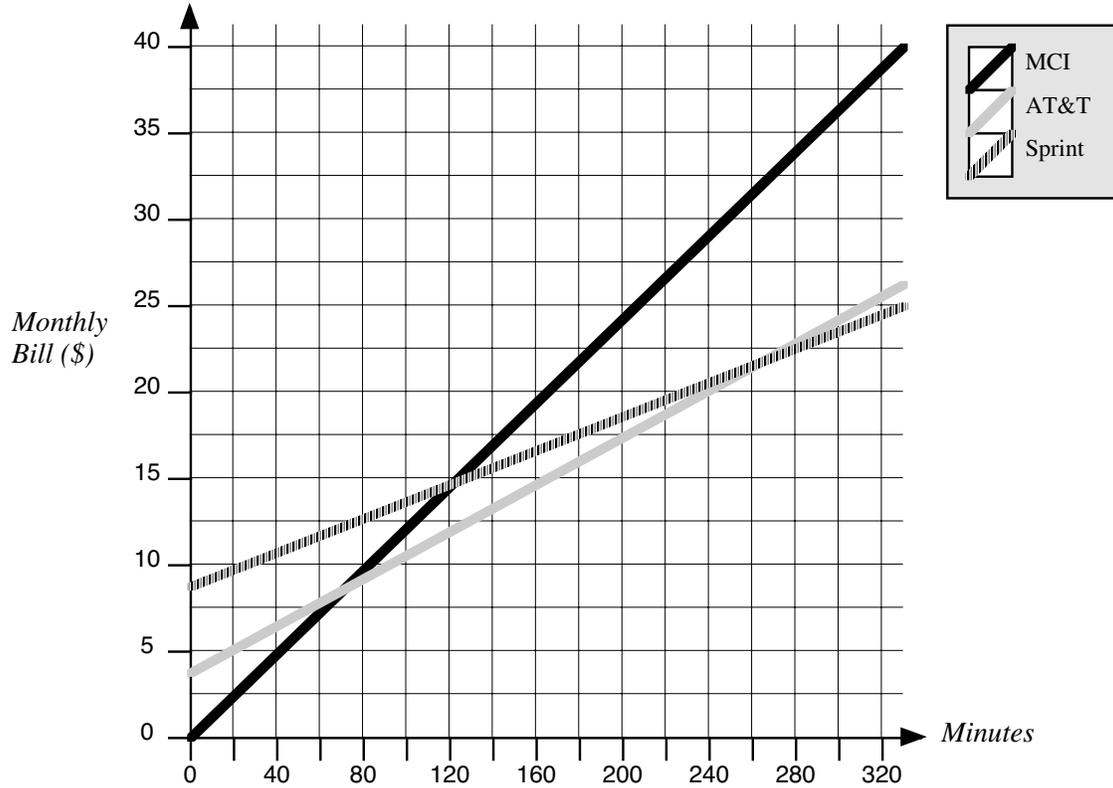
So, between 0 and 79 minutes per month, MCI is the most economical plan.

**3.(d)** From the appearance of the plot in Part (b), the Sprint plan becomes the most economical after the point where the AT&T and Sprint graphs intersect. To locate this point set the two equations equal to each other and solve for  $T$ .

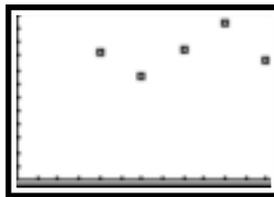
$$3.95 + 0.07 \cdot T = 8.95 + 0.05 \cdot T$$

$$\begin{aligned}
 0.02 \cdot T &= 5 \\
 T &= 5/0.02 \\
 T &= 250.
 \end{aligned}$$

So, if you use more than 250 minutes per month of long distance, the Sprint plan is the most economical.



4.(a) A plot showing the data points is given below. (The plot shown was generated on a calculator but the window size used corresponds to the sizes of the axes that were given in the problem.)



4.(b) The data appear to show two humps - one “valley” around month 6 and one “hill” around month 10. A cubic is the simplest polynomial function that can produce a graph with two humps like this.

4.(c) After entering the data into a calculator and running the “CubicReg” routine, the calculator produced the following equation for the cubic that “best fits” the data:

$$y = -0.0865 \cdot x^3 + 2.041 \cdot x^2 - 14.736 \cdot x + 41.04$$

where  $y$  represents the number of neurons (in hundreds of thousands) and  $x$  represents the month.

4.(d) Plugging  $x = 1$  into this equation gives  $y = 28.26$ . That is, 2,826,000 neurons. I think that this estimate is way too high. It is about three times the value of any of the data points. Also, I would expect the number of neurons that the bird had in January to be similar to the number that it would have in December. From the table, this would be something like 870,000 neurons.

5.(a) Let  $T =$  age of the sample in years and  $M(T) =$  amount of carbon-14 remaining in the sample (measured in micrograms). Then:

$$M(T) = 0.0001 \cdot \left(\frac{1}{2}\right)^{\frac{T}{5730}}$$

5.(b) The age of the matter will be 9240.41 years.

5.(c) Using the figure of 0.0000372 gives an age of 8174.56. This will be 1065.85 years too young.

5.(d) The correct age will be approximately 2134 years old.

6.(a) An ancient shroud was partially burnt doubling the amount of carbon-14 in it.  
**Corresponds to:** (IV)  $f(2c)$

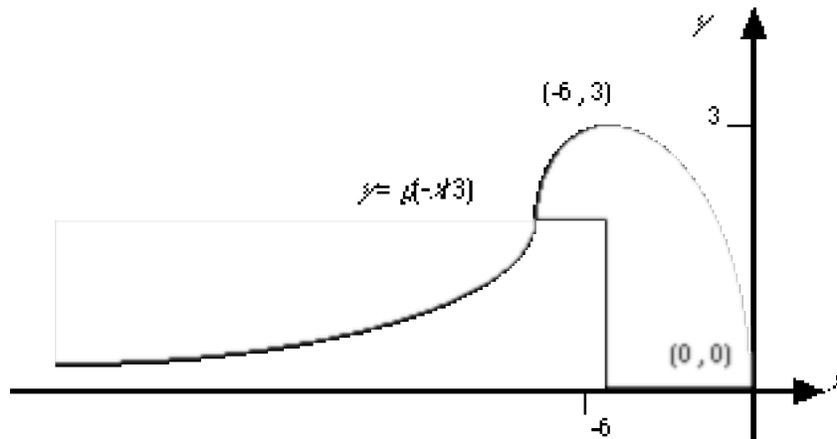
6.(b) A recently hired employee did their sums wrong and got twice the actual age of an urn.  
**Corresponds to:** (II)  $2 \cdot f(c)$

6.(c) The janitor used an antique bowl as an ashtray, and did not clean it properly, increasing the amount of carbon-14 present.  
**Corresponds to:** (V)  $f(c + 1)$

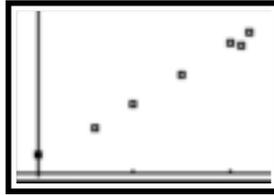
6.(d) The carbon-14 measuring machine was poorly calibrated and its readings were 25% off.  
**Corresponds to:** (III)  $f(0.75 \cdot c)$

6.(e) A professor does not accept the laboratory's results, believing a relic to be far older. He publishes his opinion as the age of the relic.  
**Corresponds to:** (I)  $f(c) + 1000$ .

7. The transformed graph is shown below.



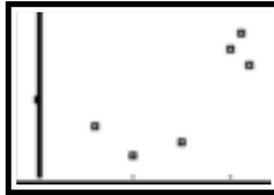
**8.(a)** The plot is shown below. To find the average expenditure per recipient, the medicaid payments were divided by the number of medicaid recipients for each year.



The plot shown above shows that the average cost of medicaid (per recipient) has been steadily rising since 1975. The costs seem to be rising in a steady fashion, with the data points lying in an almost perfectly straight line. Based on the fact that the data points lie in an almost perfectly straight line, a linear function will probably do a very good job of representing the relationship between the average cost and time.

**8.(b)** If  $T =$  years since 1975, and  $A =$  average medicaid payments (in thousands of dollars), then linear regression on a calculator gives the equation:  $A = 0.38 \cdot T + 0.93782$ .

**8.(c)** A plot of the number of recipients of medicaid versus time is given below.

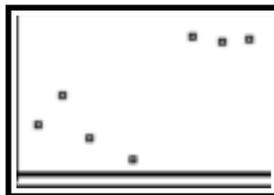


Based on the appearance of the plot, a quadratic function could do a reasonable job of representing this trend in the data.

**8.(d)** Using  $T$  as the independent variable and  $N =$  number of recipients of medicaid (in thousands) be the dependent variable, quadratic regression on a calculator gives:  $N = 6.43 \cdot T^2 - 115.67 \cdot T + 3662.18$ .

**8.(e)** The total expenditure on medicaid is the average expenditure per recipient times the total number of recipients. If  $T$  is the variable defined in part (b) and  $E$  is total expenditure on medicaid (in millions of dollars) then:  $E = A \cdot N = (0.38 \cdot T + 0.93782) \cdot (6.43 \cdot T^2 - 115.67 \cdot T + 3662.18)$ .

**8.(f)** A plot of the expenditure on medicaid (in millions of dollars) versus the number of recipients (in thousands) of medicaid is given below.



Based on this plot, it is not at all easy to say what kind of function would do a good job of representing this data. A linear function is perhaps the simplest function that you could use, and will probably do as good a job as about any other function that you could come up with.

**8.(g)** Using the symbols defined above, linear regression on a calculator gives:  $E = 22.27 \cdot N - 58459.55$ .

**8.(h)** You could compose the two functions - that is, use the output from the function in part (d) as the input to the function in part (g).

**9.(a)** Using the number of years since January 1991 as the independent variable, and the number of computers in millions as the dependent variable, the coordinates of two points from the graph are: (4.5, 10) and (9, 75).

**9.(b)** Letting  $T$  = years since January 1991, and  $C$  = number of computers (in millions), the equation for the exponential function that goes through the two points identified in part (a) is:  $C = 1.33 \cdot (1.564797)^T$ . The equation of the power function the goes through these two points is:  $C = 0.126 \cdot T^{2.90689}$ .

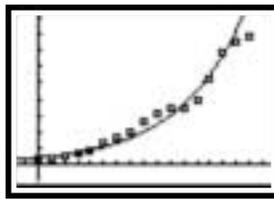
**9.(c)** Using the exponential function, there will be 1,000,000,000 computers connected to the internet in the year 2005.

**9.(d)** Using the exponential function, there were already 1,000,000,000 computers connected to the internet in the year 2000.

**9.(e)** The prediction of the researchers implies exponential growth. This is because if the number of computers increased by 10% every month, then to get the next month's number of computers, you would have to multiply the current number by a factor of 1.1. This would give an equation of the form:  $C = C_0 \cdot (1.1)^M$  where  $C_0$  is the initial number of computers connected to the internet, and  $M$  is the number of months. This is the form of an exponential equation.

**9.(f)** If the researchers are correct, then each year the number of computers connected to the internet should increase by a factor of  $(1.1)^{12} = 3.1384$ . Since the growth factor in the exponential function based on the data is only 1.564797, the researcher's prediction was for much more growth than actually took place.

**10.(a)** The plot of the data points and the graph of:  $N = 3876 \cdot (1.02068)^T$  is shown below.



**10.(b)** Based on the appearance of the graph from part (a), I would expect calculations made with values of the independent variable between  $T = 0$  and  $T = 150$  because the data points are all quite close to the graph of the exponential function in this range.

**10.(c)** There were 20,000,000 people aged 65 or older in the U.S. in 1867.

**10.(d)** According to the equation, there will be 284,000,000 people aged 65 or older in the year 2109.

**10.(e)** I would not have very much confidence in the prediction made in part (d). This is because right at the end of the plot in part (a), the data points are not going up as rapidly as the graph of the function. I would expect it to take much more time for the over 65 population to grow so large.

**10.(f)** The people who turn 65 between the years of 2010 and 2020 would have been born between 1945 and 1955. People born during this period are commonly referred to as "baby boomers" because the period of time immediately after World War 2 saw a sharp increases in the number of babies born in the US. The "surge" in the number of people turning 65 between the years of 2010 and 2020 is probably due to the "baby boomers" reaching retirement age.