Practice Problems: Test #2, Set #1

Important Information:

1. The first test will be held on **Tuesday December 3 from 7-9pm in Science Center D**.

2. The test will include approximately eight problems (each with multiple parts).

3. You will have 2 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test.

5. The specific topics that will be tested are:

- Functions defined in pieces.
- The idea of a limiting value.
- Left and right hand limits.
- Calculating limits of functions.
- Rational functions, asymptotes.
- Limits involving infinity.
- Average and instantaneous rate.
- Sketching the graph of a derivative.
- Calculating derivatives using limits.
- Graphical and verbal interpretations of the derivative.
- Short-cut rules for calculating derivatives.
- The product and quotient rules.
- Derivatives of exponential and logarithmic functions.
- Locating the maximum, minimum values of a function.

6. The problems included here have been chosen because they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

7. Good places to go for help include:

    - Office hours.
    - The labs on Tuesday 12/3.
    - The Math Question Center
    - The course-wide review on Monday evening. (Check course web site for exact time and location.)

8. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

1. Controversial state representative Arnold "Mad Dog" Johnson fears that he will not win re-election unless he pleases his wealthy constituents. Mr. Johnson introduces a bill that he says "will not penalize people with the drive and initiative to earn a good living." Specifically, Mr. Johnson's bill proposes the following scheme for state tax:

- If you earn between \$0 and \$20,000, then your state tax is 10% of your taxable income.

- If you earn between \$20,000 and \$40,000, then you pay 10% on the first \$20,000, and 5% on whatever is left over.

- If you earn over \$40,000, then you pay 10% of the first \$20,000, 5% of the next \$20,000, and nothing on the rest.

(a) Sketch a graph showing the state tax that an individual would have to pay versus taxable income.

(b) Find a collection of equations that could be used to calculate the amount of tax that an individual will have to pay, based on their taxable income. As part of your answer you should describe the domain of each equation (that is, the *x*-values that the equation is valid for).

2. In this problem the function that you are interested in will always be the function $f(x)$ defined below.

$$f(t) = \begin{cases} 2^t & ,0 < t < 1 \\ 3 - t & ,1 \le t < 2 \\ 3^t & ,2 \le t \le 3 \end{cases}$$

(a) Sketch a graph of $y = f(x)$. Make sure that you label the end-points of each portion of your graph carefully.

(b) Find the right and left hand limits of the function f as x approaches the values listed below.

- $x = 0$
- $x = 1$
- $x = 2$
- $x = 3$

(c) Describe the set of x-values where the *limit* of the function f exists. For each point between $x = 0$ and $x = 3$ (inclusive) where the *limit* of f does not exist, briefly explain why the *limit* of f does not exist there.

3. The clown loach (*Botia macracantha*) is an attractive freshwater fish from South-east Asia (see Figures 1[1] and 2[2]. It is a popular aquarium fish in the US. To meet the demand of US consumers, commercial clown loach breeders add hormones to their tanks to encourage the loaches to breed.

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[1] Image source: http://www.aquahobby.com/

[2] Image source: http://www.fishprofiles.com/

Figure 1: A clown loach (*Botia macracantha*).

Figure 2: Natural distribution of the clown loach. Most of the loaches for sale in the U.S. are bred in Borneo or Sumatra.

After an inexperienced breeder had added a dose of hormone to his tank, the number of fish in the tank, N, was quite well represented by the following function:

$$N(t) = a + 200 \cdot e^{-t},$$

where t is the number of days after adding the hormone, and a is a positive constant and e is the special number with numerical value of approximately 2.72. (Note that in this question, your answers may contain the constant a.)

(a) Using complete sentences (and possibly graphs, formulas and numbers) describe the immediate effect that the hormone had on the fish population in the tank. As part of your answer, you should determine how many fish were in the tank at $t = 0$ and describe what happened to the fish over the next few days. Be careful to explain your reasoning, and supply mathematical justifications for your answer.

(b) What will happen to the fish population in the long-term? Be careful to explain the mathematical reasoning behind your answer.

4. In this problem the function that you are interested in will always be:

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}.$$

(a) Based on the algebraic structure of the equation for f, predict the appearance of the graph of $y = f(x)$ near the point where $x = 2$. Illustrate your prediction with a sketch showing what the graph of $y = f(x)$ might look like near $x = 2$.

(b) **CAUTION:** Your success in this part of the problem will depend very heavily on your ability to:

- Enter the equation for $f(x)$ into a graphing calculator, and,
- Set the graphing window to a very specific display.

On a TI-83 calculator, the correctly entered equation will look like Figure 3(a) and the correct graphing window settings will look like Figure 3(b).

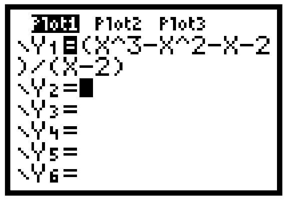

Figure 3(a): The equation entered correctly.

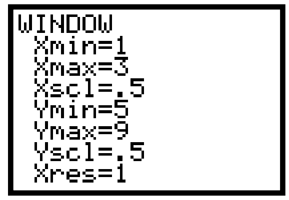

Figure 3(b): The correct graphing window.

When you have correctly set up your calculator, graph the function that you have entered. In a few sentences, describe what you saw on the calculator screen including any unusual or anomalous features.

In a sentence or two, compare the output from your calculator to the prediction that you made in Part (a).

(c) By multiplying polynomials you can show that:

$$(x-2)\cdot\left(x^2+x+1\right)=x^3-x^2-x-2.$$

You are not required to show that this equation is correct, but you are welcome to do so if you wish. What you do have to do is to use this equation to explain why the graph of $y = f(x)$ does not have a vertical asymptote at $x = 2$, and instead shows the features that you noted in Part (b). Your answer to this question should include (at minimum):

- some manipulation of the equation for the function *f*, and,
- a written description relating the algebraic structure of the equation for *f* to the appearance of the graph of $y = f(x)$.

You may also find it helpful to include graphs and numerical calculations in your answer if these enable you to make your points more clearly.

NOTE: The function *g* defined by the equation:

$$g(x) = x^2 + x + 1$$

is not the same as the function that you graphed in Part (b)[3].

5. The northernmost city in Australia is named Darwin after the famous English naturalist Charles Darwin. Darwin was the only Australian city directly attacked by the Japanese in World War 2. It is also one of the few places in the world where you have a reasonable chance of being attacked by a crocodile while walking down the main street[4]. Darwin is also the namesake of the legendary 2.25 liter beer bottles known as "Darwin Stubbies." (Two and one quarter liters is approximately equivalent to 71 fluid ounces. See Figure 4[5] for a comparison between a 40 oz bottle and a Darwin Stubbie.)

Enthusiastic (but inexperienced) male Australian youths (variously "yobbos" or "drongos") regularly attempt to drain Darwin Stubbies in order to impress their friends ("mates" or "cobbers"). Success at this

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[3] For example, you could graph $y = g(x)$ using the graphing window shown in Figure 3(b) to check.

[4] The last fatal crocodile attack on a human in Darwin was in 1971. However, crocodiles are regularly removed from the city's beaches.

[5] Image sources: http://altdotculture.com/ and http://www.kellys.com/

feat is usually followed by a loss of consciousness on the part of the youth, and various hoots of approval from his associates, including:

Figure 4: A 40 oz. bottle (left) and a Darwin Stubbie.

- “Bonza”
- “Beauty”
- “Cracker” (not to be confused with the North American usage)
- “Pearler”
- “Scorcher” or, favored in some circles,
- “Grouse.”

During one attempt that was captured on videotape, the volume of beverage left in the Darwin Stubbie after ‘t’ seconds was quite well approximated by:

$$V(t) = 1250 + (10 - t)^3,$$

where the volume is measured in milliliters (ml).

(a) How much liquid is in the bottle initially?

(b) How fast does beverage drain out of the bottle, on average, during the first 10 seconds?

(c) How fast was the liquid draining out of the bottle at the instant when the enthusiastic youth began drinking?

(d) When is the bottle completely emptied? How quickly was the fluid running out of the bottle at the instant of time immediately before the bottle was emptied?

(e) At what instant of time was the bottle being emptied the fastest? How quickly was the liquid draining out at that point in time?

(f) When people consume alcohol at a responsible rate, as much as 90% of the alcohol is broken down by an enzyme in the stomach called “alcohol dehydrogenase.” Alcohol that is broken down in the stomach does not does note lead to as profound a state of intoxication as alcohol absorbed directly into the bloodstream. When alcohol is imbibed rapidly, this first step is skipped and much higher percentage of the alcohol is absorbed into the bloodstream.

In Australia, beer normally has an alcohol content of 5% by volume. (That is, 5% of the volume of the beverage consists of alcohol.) An average person has about 4700 ml of blood[6]. The percentage of blood that is comprised of alcohol is called the Blood Alcohol Level (or BAL). On average, a person can eliminate about 15 ml of alcohol per hour[7] although this rate varies a lot depending on the individual and circumstances involved. Some typical effects of alcohol consumption are listed in Table 1[8] (see next page).

Based on the information provided in this problem, what physiological effects would you predict for a typical enthusiastic but inexperienced Australian yobbo who attempted the feat described at the beginning of this problem?

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[6] Source: Encyclopedia Brittanica, 2001.
[7] Source: http://www.intox.com/
[8] Source: K. M. Dubowski. “Stages of alcohol intoxication.” Available on-line from: www.intox.com

BAL (%)	Symptoms
0.01-0.05	Behavior normal in most subjects
0.03-0.12	Mild euphoria, sociability, talkativeness, decreased inhibitions, dimunition of judgment and control. Loss of efficiency in performance tests.
0.09-0.25	Emotional instability. Impairment of perception and memory. Increased reaction time and decrease in sensory-motor coordination. Drowsiness.
0.18-0.30	Exaggerated emotional states. Decreased muscular coordination, staggering walk and slurred speech. Disorientation and mental confusion. Vomiting.
0.25-0.40.	Inertia and loss of motor functions. Marked decrease in response to stimuli. Inability to stand or walk. Vomiting and possible incontinence. Sleep or stupor.
0.35-0.50	Coma. Depressed reflexes. Subnormal body temperature. Impairment of circulation and respiration. Incontinence. Possible death.
0.45+	Death from respiratory arrest.

Table 1.

6. In this problem, the function $f(x)$ will always refer to the function defined by the equation:

$$f(x) = \frac{1-x}{x^2},$$

with domain $x > 0$.

(a) Complete the following table. What is the value of the derivative of $f(x)$ at $x = 3$?

x	2.9	2.99	2.999	2.9999	3.0001	3.001	3.01	3.1
$\frac{f(x) - f(3)}{x - 3}$								

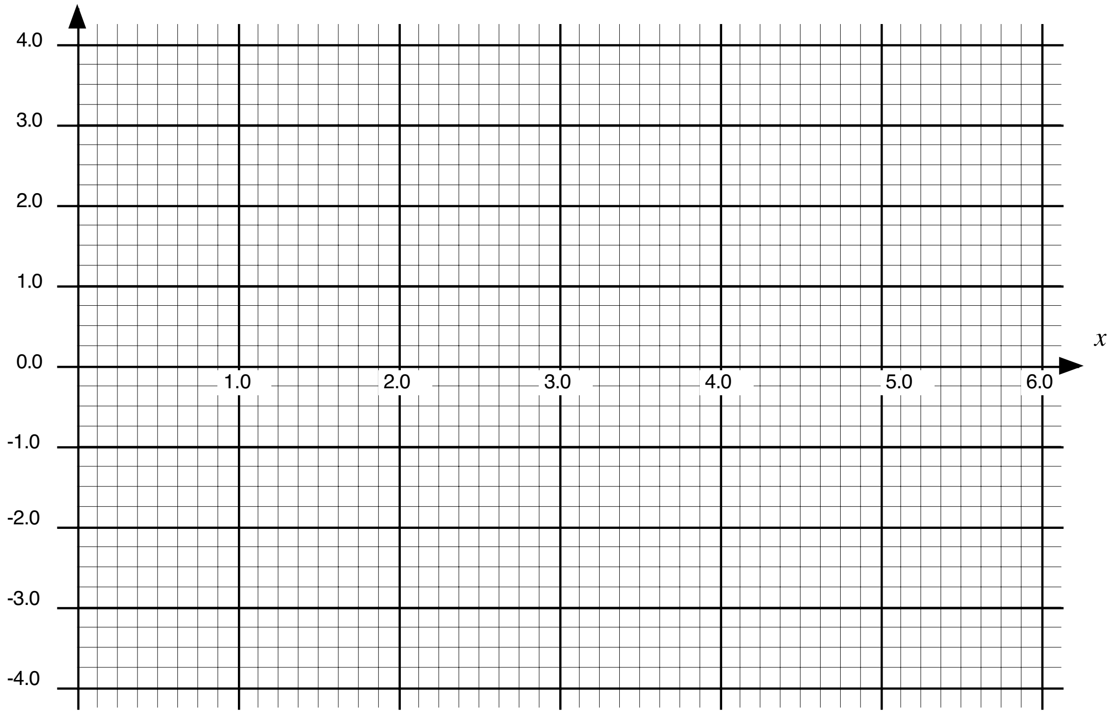
(b) Find the equation of the tangent line of $f(x)$ that is based at $x = 3$. Sketch a graph showing both $y = f(x)$ and the tangent line based at $x = 3$. Indicate how the result of Question 1 influences the appearance of your graph.

(c) Use the definition of the derivative to find an equation for the derivative of $f(x)$ at $x = a$.

(d) Use the axes provided on the next page to sketch a graph of $y = f(x)$ and directly below that, a graph of $y = f'(x)$.

(e) Explain how the features that appear on your graph of $y = f'(x)$ correspond to the features that appear on your graph of $y = f(x)$. Your answer should address the following features of the graph of $y = f'(x)$:

- intervals where $f'(x) > 0$
- intervals where $f'(x) < 0$
- points where $f'(x) = 0$
- points where $f'(x)$ attains a maximum (top of a hill) or minimum (bottom of a valley) value
- intervals where $f'(x)$ increasing
- intervals where $f'(x)$ decreasing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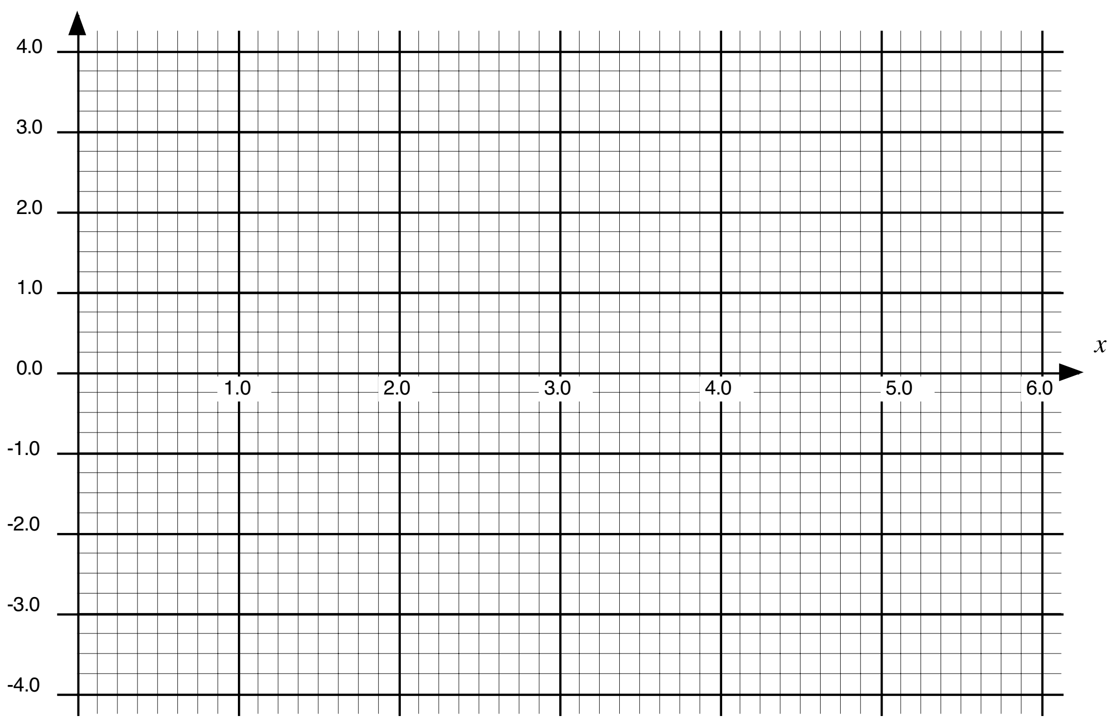
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7. The length of the day (i.e. the number of hours of daylight during a 24 hour period) varies throughout the year. During the summer, the days are long and there are many hours of daylight during each 24 hour period. During the winter, the days are short and there are fewer hours of daylight in each 24 hour period. Let the number of hours of daylight in Cambridge, Massachusetts on day 't' be represented by the function $h(t)$. (January 1 corresponds to $t = 1$ and December 31 corresponds to $t = 365$.

Some dates (valid for 2001) that might be of use when answering this question are given in Table 2 (below).

Date	Name	Event
March 20, 2001	Northern hemisphere Spring Equinox	Day and night equal length
June 21, 2001	Northern hemisphere Summer Solstice	Most hours of daylight ("longest day")
September 22, 2001	Northern hemisphere Vernal Equinox	Day and night equal length
December 21, 2001	Northern hemisphere Winter Solstice	Fewest hours of daylight ("shortest day")

Table 2

(a) Explain the meaning of the statements:

$$h(3) = 9.2$$
$$h'(62) = 0.2$$

in practical terms that would be comprehensible to a reasonably intelligent person who had no knowledge of calculus.

(b) Figure 5 (below) shows a plot of day length versus day of the year for Boston, MA. The data given in Figure 5 were collected between 12/31/97 and 12/26/98[9]

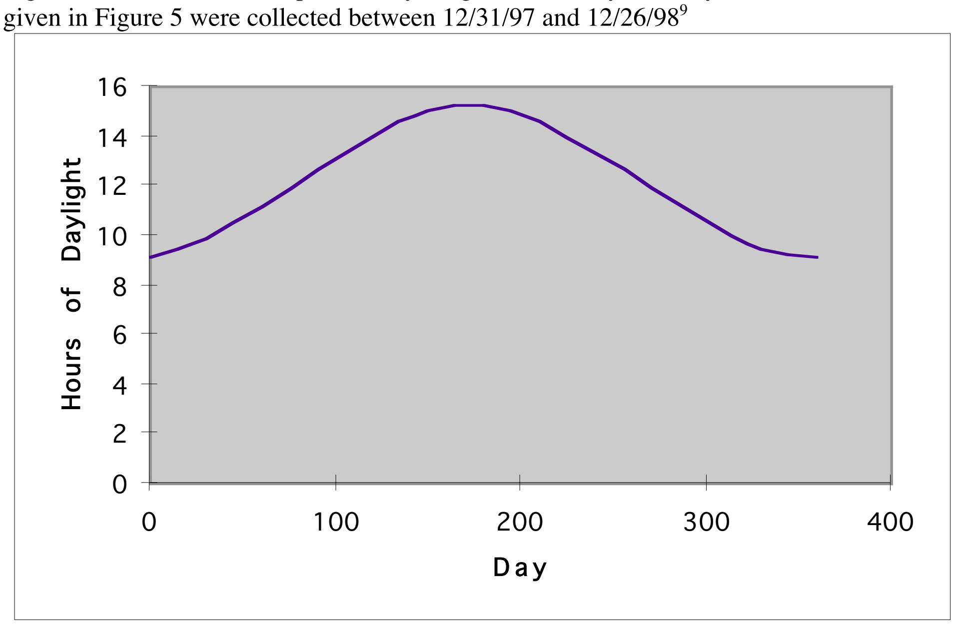

Figure 5: Hours of daylight for Boston, MA.

Use Figure 5 to graphically represent the two pieces of information given in Part (a). (Possibilities include: heights on the graph at certain points, slopes of tangent lines at certain points, slopes of secant lines at certain points, etc.)

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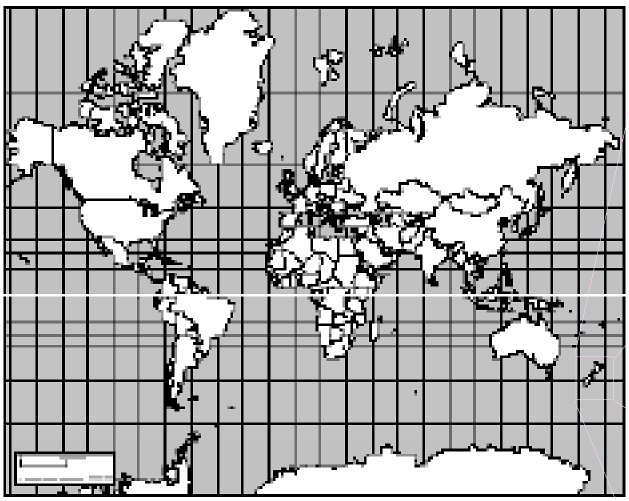
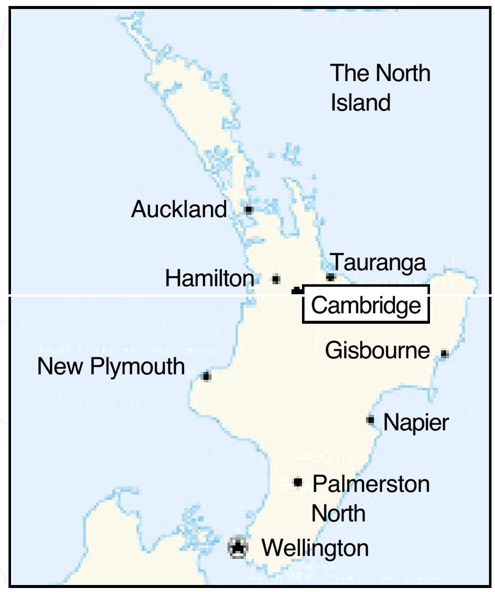
[9] Source: The Texas Education Network, Texas Essential Knowledge and Skills, Mathematics Module: Oscillations.

(c) What can you say about the value of $h'(t)$ when t is in the month of May? (You should comment on whether $h'(t)$ is positive or negative, and whether $h'(t)$ is increasing or decreasing.) Explain your reasoning.

(d) What can you say about the value of $h'(t)$ when t is in the month of September? (You should comment on whether $h'(t)$ is positive or negative, and whether $h'(t)$ is increasing or decreasing.) Explain your reasoning.

(e) If $h(t)$ applied to Cambridge, New Zealand (See Figures 6[10] and 7 [11]), instead of Cambridge, Massachusetts how would your answers to Parts (c) and (d) be different? Explain your reasoning.





Figures 6 amd 7: Location of New Zealand in the Southern Hemisphere. Principal cities of the North Island of New Zealand.

8. In this problem, the function f will always refer to the function defined by the equation:

$$f(x) = \frac{1-x}{1-x^2}$$

and the function g will always refer to the function defined by the graph shown on the next page. In each of the following cases decide whether or not the limit (possibly a left- or a right-hand limit, check the notation carefully) exists. If you believe that a limit exists, determine its value. If you believe that a limit does not exist, give your reason.

(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 0} g(x)$

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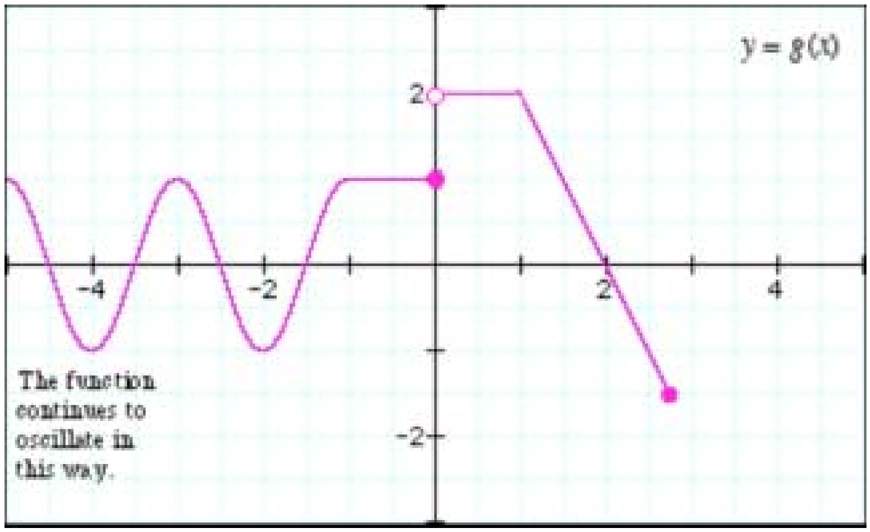
[10] Image source: Cartographic Research Lab, University of Alabama. http://alabamamaps.ua.edu/world/world/

[11] Image source: CIA World Factbook, 2001. Available on-line from http://www.cia.gov/cia/publications/factbook/

(c) $\lim_{x \to -\infty} g(x)$

(d) $\lim_{x \to 2^+} \frac{f(x)}{g(x)}$



9. Iraqi dictator Saddam Hussein is occasionally shown on television firing a rifle into the air to celebrate an event. Rob Neel is an enthusiastic but inexperienced non-Australian youth who was deeply impressed by the Iraqi dictator's methods of celebration and decided to emulate them himself.

In this problem, T will always represent the number of seconds since Rob Neel fired his rifle. In this problem, $d(T)$ will always represent the height (in feet) of the bullets above the ground. The function d is defined by the equation given below.

$$d(T) = -16 \cdot T^2 + 2367 \cdot T + 5.$$

(a) The *velocity* of the bullets is given by the derivative of $d(T)$. Find an equation for the velocity of the bullets.

(b) How long did it take the bullets to reach their maximum height? How high did the bullets go?

(c) The equation defining $d(T)$ is a quadratic equation in standard form. Use the technique of *completing the square* to convert the equation for $d(T)$ into vertex form.

10. In this problem, the function $f(x)$ will always refer to the function defined by the following equation:

$$f(x) = x^3 - 5x + 1.$$

Find an equation for the derivative function $f'(x)$. (Note that you do *not* have to use the limit definition of the derivative here.) Use the equation for $f'(x)$ to find the places where f has a local maximum (top of a hill) or local minimum (bottom of a valley). Briefly explain your reasoning process.