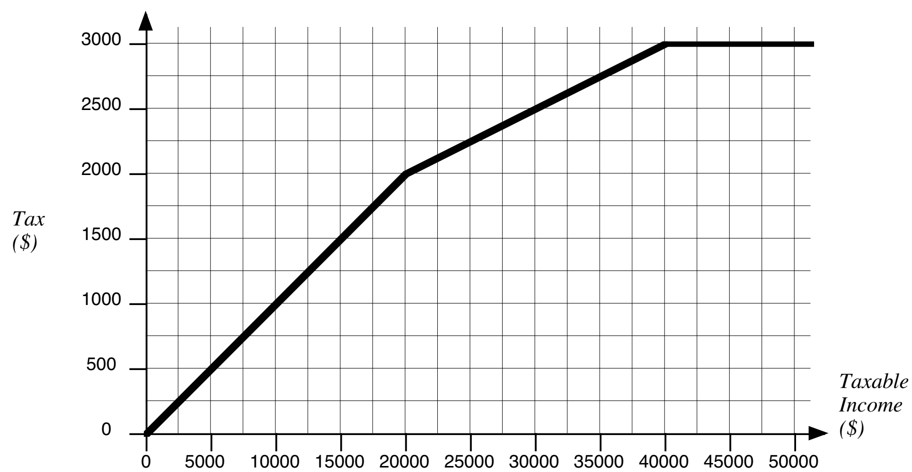
Solutions: Test #2, Set #1

These answers are provided to give you something to check your answers against and to give you some idea of how the problems were solved. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.

1.(a) The graph of tax paid as a function versus taxable income under Mr. Johnson's controversial plan is shown in the graph below.



1.(b) Let I = taxable income in dollars be the independent variable and T = tax owed (in dollars) be the dependent variable. Then the relationship between taxable income and tax owed can be represented by the collection of equations given in the table below.

Domain equation valid for	Equation
0 ≤ I ≤ 20,000	T = 0.1*I
20,000 < I ≤ 40,000	T = 2000 + 0.05*(I – 20,000)
40,000 < I	T = 3000.

Using the more conventional notation for functions defined in pieces, this function could be written out as:

$$T(I) = \begin{cases} 0.1*I & , 0 \le I \le 20{,}000 \\ 2000 + 0.05*(I - 20{,}000) & , 20{,}000 < I \le 40{,}000 \\ 3000 & , 40{,}000 < I \end{cases}$$

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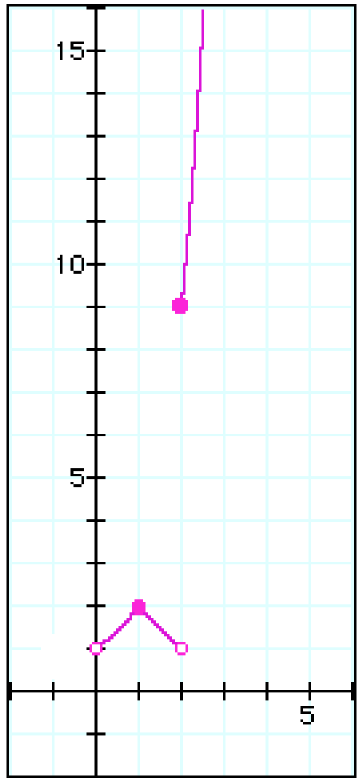
2.(a) The graph of $y = f(x)$ is shown below.



2.(b) The left and right hand limits of the function f are given in the table below.

x	Left hand limit of the function f	Right hand limit of the function f
0	Does not exist	1
1	2	2
2	1	9
3	27	Does not exist

Note that f does not have a left hand limit at $x = 0$ because f is not defined when $x \leq 0$. Similarly, f does not have a right hand limit at $x = 3$ because f is not defined when $x > 3$.

2.(c) The answers that you get here will depend a lot on what you think it means for a *limit* to exist at a finite value of x. The official mathematical definitions are the following:

• If the x-value in question is the **left end point of an interval** then the *limit* exists provided that the right hand limit of the function exists there. The value of the *limit* is equal to the value of the right hand limit.

• If the x-value in question is the **right end point of an interval** then the *limit* exists provided that the left hand limit of the function exists there. The value of the *limit* is equal to the value of the left hand limit.

• If the x-value in question is not an end-point, then the *limit* exists at the x-value in question provided that the left hand limit and the right hand limit are equal.

Based on these criteria, the *limit* of f exists at every x-value from $x = 0$ to $x = 3$ (inclusive) with the exception of $x = 2$. The reason that the *limit* of f does not exist at $x = 2$ is that the left hand limit (= 1) and the right hand limit (= 9) are not equal.

Note: If you did not know about the convention for limits and end-points at intervals, you would also be justified in believing that the *limit* of f did not exist at either $x = 0$ or at $x = 3$ because at these two points one of the left hand or right hand limits does not exist.

3.(a) The initial fish population can be calculated by substituting $t = 0$ into the equation,

$$N(t) = a + 200 \cdot e^{-t}.$$

This gives an initial fish population of $a + 200$ fishes. As t increases, the number of fish in the tank will **decrease**. You can see that this is what will happen by examining the behavior of the equation for $N(t)$ term by term.

- First term, a: This is a constant and will be equal to a regardless of what the value of t is.

- Second term, $200 \cdot e^{-t}$: Re-writing this without the negative sign in the exponent gives:

$$200 \cdot e^{-t} = 200 \cdot \left(\frac{1}{e}\right)^t.$$

The numerical value of the special number e is about 2.72. The really important fact about the special number e is that it is greater than one, as this means that the second term of $N(t)$ will be a decreasing exponential function. Therefore, overall,

$N(t)$ = (constant) + (decreasing exponential function)

so in the short term, the numbers of fish in the tank will decrease from their initial level of $200 + a$ fishes.

3.(b) In the long term, the first term of $N(t)$ will stay constant, and the second term of $N(t)$ will get closer and closer to zero. Therefore, in the long term, $N(t)$ will resemble:

$N(t)$ = (constant) + (quantity getting closer and closer to zero)

so in the long term, the fish population will decrease until there were approximately a fish left in the tank. The population should stabilize at this level.

4.(a) There are two reasonably sensible predictions that you could make about the appearance of the graph of $y = f(x)$ near $x = 2$. Recall that f is the function defined by the equation:

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}.$$

Either of these predictions is acceptable, so long as it is accompanied by:

(I) An analysis of the algebraic structure of the equation for $f(x)$, and,
(II) A graph that is consistent with that analysis.

- **Prediction 1:** Notice that the denominator of the equation for $f(x)$,

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}$$

includes the factor ($x - 2$). This means that when x is near 2, evaluating $f(x)$ will require division by a very small number that is close to zero. When you divide a quantity by a number that is very close to zero, it makes the result very large overall. Based on this analysis, I would predict that the graph of $y = f(x)$ should show a vertical asymptote at $x = 2$.

Based on this prediction, the graph of $y = f(x)$ could resemble any of the following.

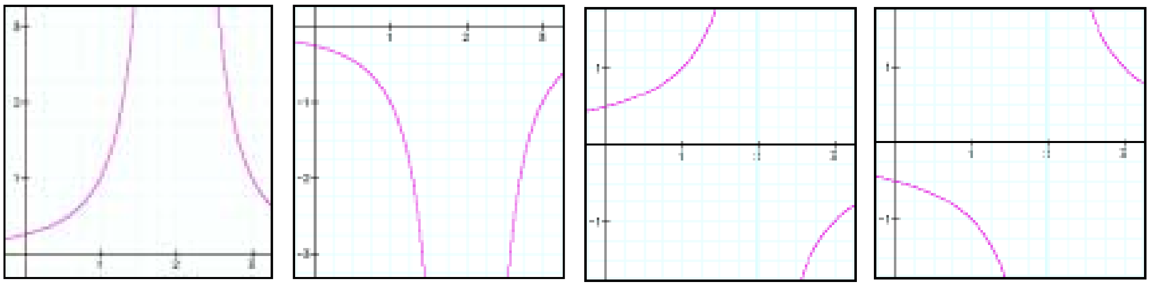


Figure 1(a, b, c, d): Possible appearances of the graph of $y = f(x)$ near $x = 2$, based on the prediction that $y = f(x)$ will have a vertical asymptote at $x = 2$.

- **Prediction 2:** Notice that the denominator of the equation for $f(x)$,

$$f(x) = \frac{x^3 - x^2 - x - 2}{x - 2}$$

includes the factor ($x - 2$). This means that when x is near 2, evaluating $f(x)$ will require division by a very small number that is close to zero. However, also note that when you substitute $x = 2$ into the numerator of the equation for $f(x)$, you get zero. Therefore, when x is near 2, both the numerator and the denominator of the equation for $f(x)$ are both close to zero. Ordinarily, when you divide by something that is very close to zero, it makes the quotient very large overall. However, when the numerator of the quotient is also a tiny number, the result of dividing by something close to zero may not be too large. Based on this reasoning, I would expect that the graph of $y = f(x)$ would have a hole in it near $x = 2$, but that there would be no vertical asymptote.

Based on this prediction, the graph of $y = f(x)$ could resemble the following.

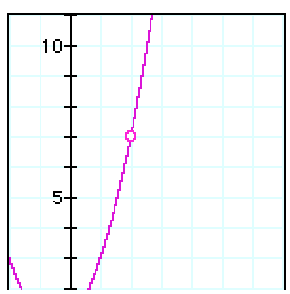


Figure 2: Possible appearance of $y = f(x)$ based on prediction of no vertical asymptote near $x = 2$.

4.(b) The graph from a TI-83 is shown below. This graph clearly shows that f is not defined at $x = 2$ as there is a hole in the graph there. There is no vertical asymptote at $x = 2$. This situation appears to correspond to the second prediction made above - that the effect of the small numerator will compensate for the small denominator, and the graph of $y = f(x)$ will not have a vertical asymptote at $x = 2$.

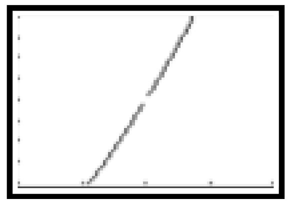


Figure 3: Graph of $y = f(x)$ from TI-83.

4.(c) Using the fact that:

$$(x-2)\cdot\left(x^2+x+1\right) = x^3 - x^2 - x - 2$$

you can re-write the equation for $f(x)$ as follows:

$$f(x) = \frac{x^3 - x^2 - x - 2}{x-2} = \frac{(x-2)\cdot\left(x^2+x+1\right)}{(x-2)} = \frac{(x-2)}{(x-2)}\cdot\left(x^2+x+1\right).$$

When x is near 2, the value of $f(x)$ will be approximately equal to:

$$f(x) = \frac{(x-2)}{(x-2)}\cdot\left(x^2+x+1\right) \approx 1\cdot\left(2^2+2+1\right) = 7\,.$$

So, as x approaches 2 from either the left or the right, the value of $f(x)$ will be close to 7. In terms of the appearance of the graph, this means that instead of having vertical asymptotes that shoot up to $+\infty$ or plunge down to $-\infty$, the graph of $y = f(x)$ will simply get close to a height of 7 as x gets close to a value of 2. (See the values of $f(x)$ obtained by tracing the graph below.)

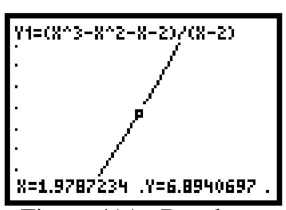


Figure 4(a): Result of tracing on the graph of $y = f(x)$ slightly to the left of $x = 2$.

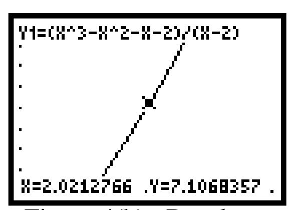


Figure 4(b): Result of tracing on the graph of $y = f(x)$ slightly to the right of $x = 2$.

5.(a) $V(0) = 2250$ ml.

5.(b) The average rate of change of the function $V(t)$ between $t = 0$ and $t = 10$ is given by:

$$\frac{V(10)-V(0)}{10-0}=\frac{1250-2250}{10}=-100 \text{ ml per second.}$$

5.(c) The derivative of the function $V(t) = 1250 + (10 - t)^3$ can be found by FOILing out the bracket and then differentiating what you get. If you do this, then after FOILing you will have:

$$V(t) = 2250 - 300t + 30t^2 - t^3.$$

Taking the derivative of this gives:

$$V'(t) = -300 + 60t - 3t^2$$

Substituting $t = 0$ into this expression gives $V'(0) = -300$ ml per second.

5.(d) To find when the bottle was completely empty, you need to solve the equation $V(t) = 0$ for t. Doing this gives that the bottle was empty after 20.77 seconds. Substituting $t = 20.77$ into the equation for the derivative gives $V'(20.77) = -348.12$ ml per second.

5.(e) The bottle will empty the fastest when the absolute value of $V'(t) = -300 + 60t - 3t^2$ is the largest on the closed interval [0, 20.77]. If you take the derivative of $V'(t) = -300 + 60t - 3t^2$ (that is, you take the derivative of the derivative), set it equal to zero and solve for t you get $t = 10$. However, this is when the bottle was draining the slowest. To determine when the bottle was draining the fastest, we have to look at the absolute values $V'(t) = -300 + 60t - 3t^2$ at the endpoints of the closed interval – that is, at $t = 0$ and at $t = 20.77$. Based on the answers to Parts (c) and (d), the bottle was draining the fastest at $t =$ 20.77 seconds.

5.(f) The Darwin Stubby contained $0.05 \cdot 2250 = 112.50$ ml of alcohol. If all of that alcohol went into the person's blood stream, then the blood alcohol concentration would be 112.5/(4700 + 112.5) = 0.0234. This is equal to a blood alcohol level of 2.34% which would be sufficient to put the person into a coma and induce death from respiratory arrest. Don't try this at home, even if you are an Australian.

6.(a) The completed table is given below.

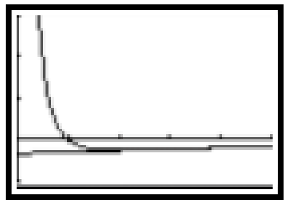
x	2.9	2.99	2.999	2.9999	3.0001	3.001	3.01	3.1
$\frac{f(x)-f(3)}{x-3}$	0.03699	0.037037	0.037037	0.037037	0.037037	0.037037	0.037037	0.03699

Based on the trend of the values in the table, the limiting value of $\dfrac{f(x)-f(3)}{x-3}$ as x approaches 3 appears to be equal to. Therefore the value of the derivative of $f(x)$ at $x = 3$ is approximately 0.037037.

6.(b) The slope of the tangent line is 0.037037, and the tangent line passes through the point $(3, f(3)) =$ (3, -2/9). The equation of the line with slope 0.037037 that passes through the point (3, -2/9) is:

$$y = 0.037037x - 0.3333332222.$$

The graph of $y = f(x)$ and the graph of the tangent line is shown below.



6.(c) There are four steps that one normally needs to complete in order to calculate the derivative algebraically:

(a) Use the definition of the derivative to correctly find $f(a + h)$.

(b) Use the definition of the function to calculate the difference quotient,

$$\frac{f(a+h)-f(a)}{h}.$$

(c) Simplify the difference quotient as much as possible. A good clue that you have done this is that there should be no factors of h remaining in the denominator of the difference quotient.

(d) Find the limiting value of the difference quotient as $h \to 0$.

Performing these steps, one after the other.

(a) Formulating $f(a + h)$:

$$f(x) = \frac{1-(a+h)}{(a+h)^2}.$$

(b) Calculating the difference quotient:

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1-(a+h)}{(a+h)^2} - \frac{1-a}{a^2}}{h}.$$

(c) Simplifying the difference quotient gives:

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1-(a+h)}{(a+h)^2} - \frac{1-a}{a^2}}{h} = \frac{\frac{a^2 - a^2 \cdot (a+h) - (a+h)^2 + a \cdot (a+h)}{a^2 \cdot (a+h)^2}}{h} = \frac{a^2 h - 2ah - h^2 + ah^2}{ha^2(a+h)^2}.$$

and simplifying further gives:

$$\frac{f(a+h)-f(a)}{h} = \frac{a^2 - 2a - h + ah}{a^2(a+h)^2}.$$

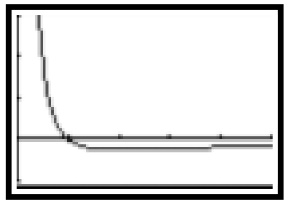
Note that the denominator of the difference quotient no longer contains any *factors* of h. There is an h still in the denominator, but it is not a factor by itself.

(d) Taking the limit as $h \to 0$ gives that the derivative of $f(x)$ at $x = a$ is equal to:

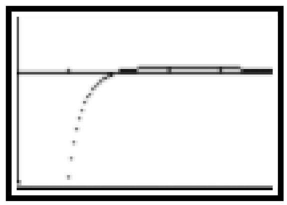
$$f'(a) = \frac{a^2 - 2a}{a^4} = \frac{1}{a^2} - \frac{2}{a^3}.$$

6.(d) The two graphs ($y = f(x)$ and $y = f'(x)$) are shown below

Graph of $y = f(x)$:



Graph of $y = f'(x)$:



6.(e) The correspondences between the behavior of the original function and the behavior of the derivative are summarized in the table shown below.

Behavior of original function/appearance of graph of y = f(x)	Behavior of derivative/appearance of graph of $y = f'(x)$
Increasing (as graph read from left to right)	Derivative is positive
Decreasing (as graph read from left to right)	Derivative is negative
Top of hill (maximum value) or bottom of valley (minimum value)	Derivative is zero
Concave up	Derivative is increasing
Concave down	Derivative is decreasing
Inflection point (concavity changes)	Derivative graph has a maximum value (top of a hill) or a minimum value (bottom of a valley)

For the specific example of $f(x) = \frac{1-x}{x^2}$:

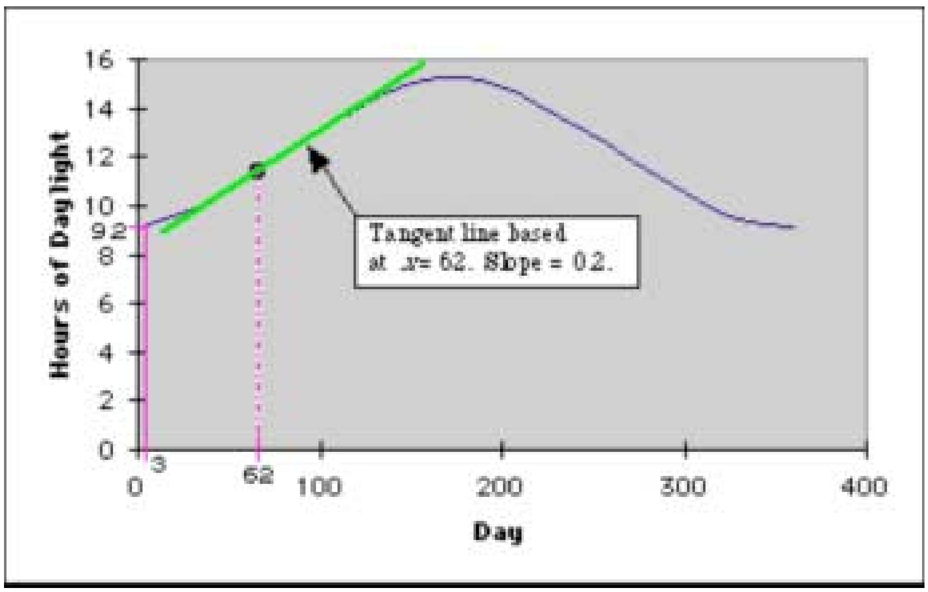
Original Function	Derivative	Points or intervals
Increasing	Positive	2 < x
Decreasing	Negative	0 < x < 2
Max. or min.	Zero	x = 2
Concave up	Increasing	0 < x < 3
Concave down	Decreasing	3 < x
Inflection point	Max. or min.	x = 3

7.(a) The practical interpretations of the symbolic statements are given below.

$h(3) = 9.2$ means that on January 3, Cambridge, MA will experience 9.2 hours of daylight.

$h'(62) = 0.2$ means that the on 63rd day of the year, Cambridge MA will have approximately 0.2 hours (i.e. 12 minutes) more daylight than on 62nd day of the year. For example, if there were 10 hours of daylight in Cambridge MA on the 62nd day of the year, then Cambridge MA would have approximately 10.2 hours of daylight on the 63rd day of the year.

7.(b) The graphical representations of the symbols are shown below. In words, $h(3) = 9.2$ means that when the independent variable is equal to 3, the dependent variable is equal to 9.2. Furthermore, $h'(62) = 0.2$ means that the slope of the tangent line is 0.2 (when the tangent line is drawn at the point the dependent variable is equal to 62).



7.(c) Between the Spring Equinox and the Summer Solstice, I would expect $h(t)$ to be an increasing, concave down function. Throughout this period, the derivative $h'(t)$ will be positive as $h(t)$ is increasing. Furthermore, the derivative $h'(t)$ will itself be decreasing as the function $h(t)$ is concave down. As May lies between the Spring Equinox (March) and the Summer Solstice (June), then throughout May $h'(t)$ should be positive and decreasing.

7.(d) Between the Summer Solstice and the Vernal Equinox, I would expect the function $h(t)$ to be a decreasing, concave down function. Between the Vernal Equinox and the Winter Solstice, I would expect $h(t)$ to be a decreasing, concave up function. Thus, during the entire month of September, the derivative $h'(t)$ will be negative to reflect the fact that $h(t)$ is decreasing. The concavity is a little more complicated, as the Vernal Equinox occurs on September 22. So, between September 1 and September 22, the derivative $h'(t)$ will be decreasing as $h(t)$ is concave down during this time period. However, from September 22 to September 30 the derivative $h'(t)$ will be increasing as $h(t)$ is increasing during this period.

7.(e) New Zealand is in the Southern hemisphere, and so the seasons occur at the opposite times, compared with the Northern hemisphere. For example, when it is summer in the Northern hemisphere, it is winter in the Southern hemisphere, etc. This means that if $h(t)$ referred to Cambridge, New Zealand rather than Cambridge, MA, then the behavior of the derivative $h'(t)$ would be the opposite of that described in the answers to Parts (c) and (d). For example, during May, $h'(t)$ would be negative and increasing.

8.(a) $\lim_{x \to 1} f(x) = \frac{1}{2}$.

To deduce this limit, note that the denominator of *f* can be factored:

$$f(x) = \frac{1-x}{1-x^2} = \frac{1-x}{(1-x)\cdot(1+x)} = \frac{1-x}{1-x}\cdot\frac{1}{1+x}.$$

When x is close to 1, $f(x)$ will be very close to (1)·(1/2) = 1/2.

8.(b) $\lim_{x \to 0} g(x)$ does not exist because the left hand limit (which is equal to 1) is not equal to the right hand limit (which is equal to 2).

8.(c) $\lim_{x \to -\infty} g(x)$ does not exist because the function *g* oscillates up and down, never settling down to approach a steady height.

8.(d) $\lim_{x \to 2^+} \frac{f(x)}{g(x)} = -\infty$.

To deduce this, you can look at the behavior of the functions *f* and *g* as each approach $x = 2$ from the right, and then combine these to understand how the quotient $\frac{f(x)}{g(x)}$ behaves as x approaches 2 from the right.

Behavior of $f(x)$ as x approaches 2 from the right.

The function *f* is defined when $x = 2$. Its value (which is equal to the limit from the right) is: $f(2) = 1/3$. Therefore, when x is a number slightly larger than 2, $f(x)$ will be approximately equal to 1/3.

Behavior of $g(x)$ as x approaches 2 from the right.

Inspection of the graph of *g* shows that when x is a number slightly larger than 2, $g(x)$ is a negative number that is very, very close to zero.

Behavior of $\frac{f(x)}{g(x)}$ as x approaches 2 from the right.

Combining the previous two observations, you have that when x is a number slightly greater than 2,

$$\frac{f(x)}{g(x)} = \frac{\approx \frac{1}{3}}{-(small)} = -(big).$$

So, as x gets closer and closer to 2 from the right, $\frac{f(x)}{g(x)}$ is a larger and larger negative number.

9.(a) As you learned in the lab with the motion detector, *velocity* is the derivative of the distance function. Therefore, you can obtain the velocity of the bullets by finding the derivative of $d(T)$:

$$Velocity = d'(T) = -32 \cdot T + 2367.$$

9.(b) When the bullets will reach maximum height, their velocity will equal zero. You can calculate when this occurs by finding the value of T for which $d'(T) = 0$. Solving this equation for T:

$$T = \frac{2367}{32} \approx 73.97 \text{ seconds after the rifle was fired.}$$

The maximum height that the bullets achieved can be found by substituting $T = 73.97$ into the equation for the function $d(T)$ (NOT the derivative). This gives:

$$\text{Maximum height} = d(73.97) \approx 87547.02 \text{ feet.}$$

9.(c) The process of completing the square for $d(T)$ is shown below. When completing the square you are ultimately trying to convert a quadratic equation into *vertex form* which resembles:

$$y = c \cdot (x - h)^2 + k,$$

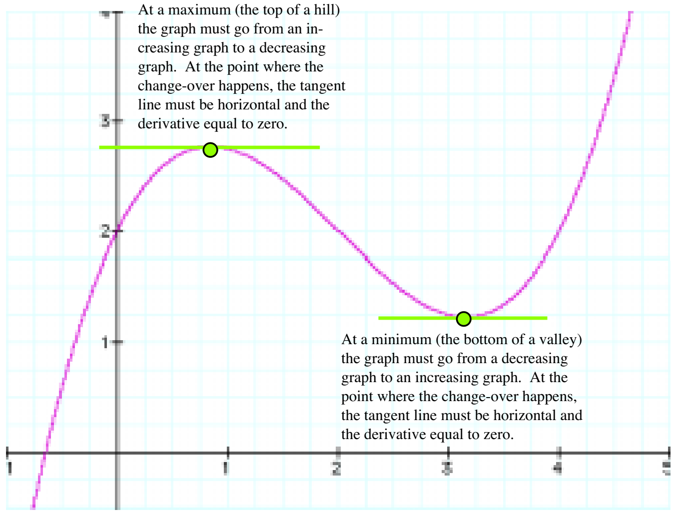
where c, h and k are constants.

$$\begin{aligned}
d(T) &= -16 \cdot T^2 + 2367 \cdot T + 5 \\
&= -16 \cdot \left[T^2 - \tfrac{2367}{16} \cdot T - \tfrac{5}{16}\right] \\
&= -16 \cdot \left[T^2 - \tfrac{2367}{16} \cdot T + \left(\tfrac{2367}{32}\right)^2 - \left(\tfrac{2367}{32}\right)^2 - \tfrac{5}{16}\right] \\
&= -16 \cdot \left[\left(T - \tfrac{2367}{32}\right)^2 - \left(\tfrac{2367}{32}\right)^2 - \tfrac{5}{16}\right] \\
&= -16 \cdot \left(T - \tfrac{2367}{32}\right)^2 + 16 \cdot \left(\tfrac{2367}{32}\right)^2 + 5 \\
&= -16 \cdot \left(T - \tfrac{2367}{32}\right)^2 + 87547.02.
\end{aligned}$$

10.(a) Using the short-cut rules for differentiation, $f(x) = x^3 - 5x + 1$ can be differentiated without all of the steps and manipulations that the limit definition requires. The derivative of f is given by the equation:

$$f'(x) = 3x^2 - 5.$$

To locate the places where the graph of the original function $y = f(x)$ has a hill-top (maximum) or valley-bottom (minimum) value, you can set the derivative equal to zero and solve for x. (The reason that this technique works is shown below.)



Solving the equation $f'(x) = 0$ for *x*:

$$3x^2 - 5 = 0$$

$$3x^2 = 5$$

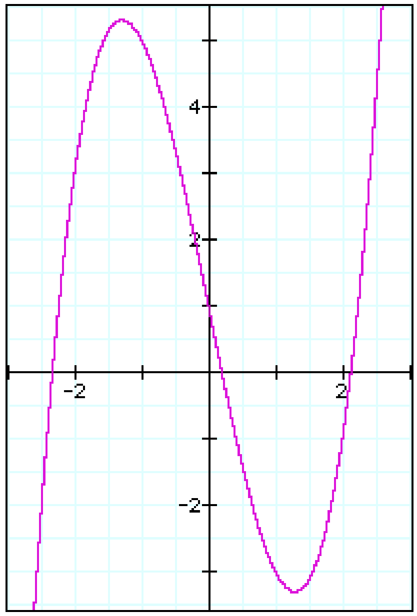
$$x^2 = \frac{5}{3}$$

$$x = \pm\sqrt{\frac{5}{3}}.$$

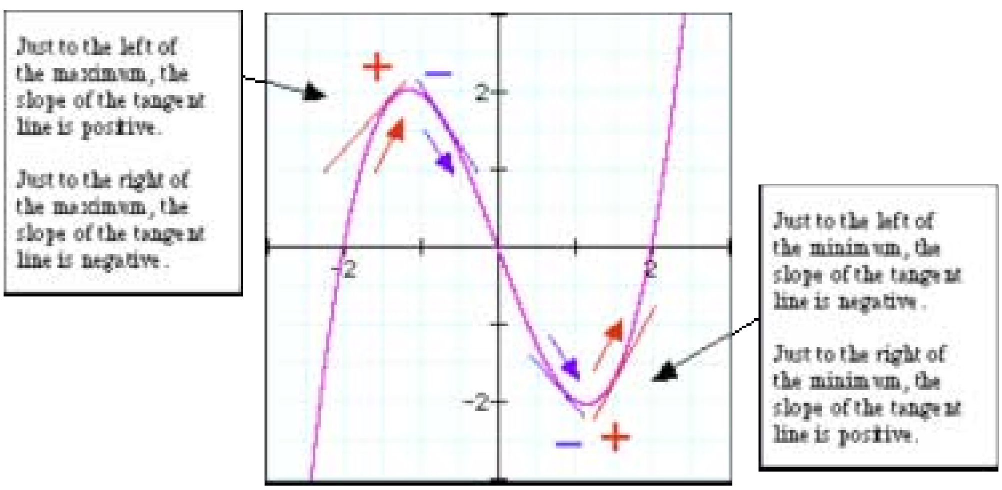
This gives the *x*-coordinates of the hill-tops (maximums) and valley-bottoms (minimums). To find the *y*-coordinates of these points, you can substitute each value of *x* into the equation for *f*(*x*). This gives the *x* and *y*-coordinates of each point:

- $(-\sqrt{\tfrac{5}{3}}, 5.5033)$
- $(\sqrt{\tfrac{5}{3}}, -3.3033)$.

To determine whether a given point is a hill-top (maximum) or valley-bottom (minimum) you can locate the points on the graph of *y* = *f*(*x*) (see below).



If a graph of $y = f(x)$ is not readily available, you can also check the sign of the derivative just to the left and just to the right of the point where $f'(x) = 0$. The pattern of "+" and " –" can reveal whether the point is a maximum or a minimum (see diagram below).



Performing the calculations for the points obtained here gives the following:

x-coordinate	$f'(x)$ just to the left of the point	$f'(x)$ just to the right of the point	Type of Point
$x = -\sqrt{\frac{5}{3}}$	+0.07	–0.0077	Hill-top (Maximum)
$x = \sqrt{\frac{5}{3}}$	–0.0077	0.07	Valley-bottom (Minimum)