# Practice Problems: Test #2, Set #2

Important Information:

1. The first test will be held on **Tuesday December 3 from 7-9pm in Science Center D**.

2. The test will include approximately eight problems (each with multiple parts).

3. You will have 2 hours to complete the test.

4. You may use your calculator and one page (8" by 11.5") of notes on the test.

5. The specific topics that will be tested are:

   - Functions defined in pieces.
   - The idea of a limiting value.
   - Left and right hand limits.
   - Calculating limits of functions.
   - Rational functions, asymptotes.
   - Limits involving infinity.
   - Average and instantaneous rate.
   - Sketching the graph of a derivative.
   - Calculating derivatives using limits.
   - Graphical and verbal interpretations of the derivative.
   - Short-cut rules for calculating derivatives.
   - The product and quotient rules.
   - Derivatives of exponential and logarithmic functions.
   - Locating the maximum, minimum values of a function.

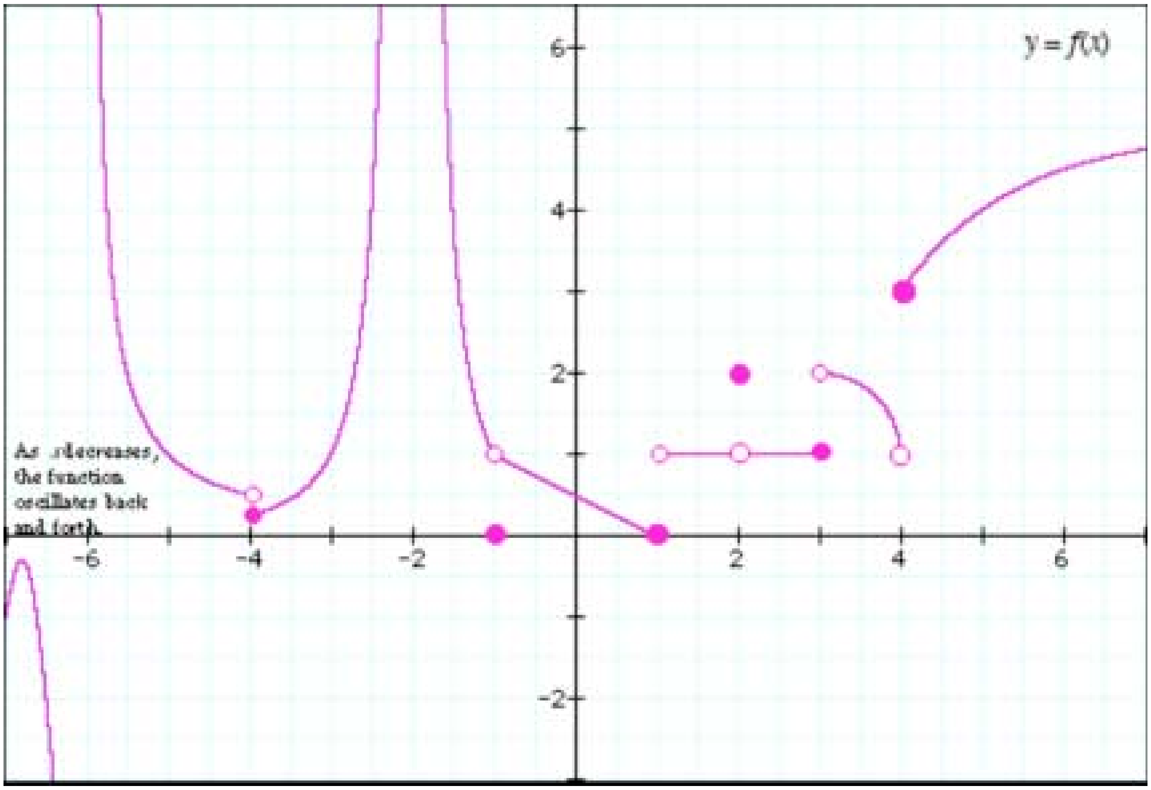
6. The problems included here have been chosen because they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

7. Good places to go for help include:

   - Office hours.
   - The labs on Tuesday 12/3.
   - The Math Question Center
   - The course-wide review on Monday evening. (Check course web site for exact time and location.)

8. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

**1.** In this problem, the function *f* is the function defined by the graph shown below.




Evaluate the following limits. If the limit does not exist, explain why.

**(a)** $\lim_{x \to +\infty} f(x)$

**(b)** $\lim_{x \to -\infty} f(x)$

**(c)** $\lim_{x \to -4} f(x)$

**(d)** $\lim_{x \to -6} f(x)$

**(e)** $\lim_{x \to -1} f(x)$

**(f)** $\lim_{x \to 1} f(x)$

**(g)** $\lim_{x \to 2} f(x)$

**(h)** $\lim_{x \to 2} f(2)$

**(i)** $\lim_{x \to -2} f(x)$

**2.** A car starts moving down a straight road at $t = 0$. After traveling for $t$ seconds, the car has covered $s(t)$ feet. The function $s$ is defined by the equation given below.

$$s(t) = \begin{cases} 2.64 \cdot t^2 & ,0 \le t \le 20 \\ 105.6t + 1056 & ,20 < t \end{cases}$$

**(a)** What is the average speed over the interval [0, 10] ?

**(b)** What will the speedometer of the car read at the instant of time when $t = 10$ ?

**(c)** What will the speedometer of the car read at the instant of time when $t = 30$ ?

**(d)** What is the average speed of the car over the interval $[10, 10 + k]$ where $k$ is a number between 0 and 10?

**(e)** What is the limit of the average speed of the car over the interval $[10, 10 + k]$ as $k \to 0^+$?

**3.** In this problem the function $f(x)$ will always refer to the function defined by the equation given below.

$$f(x) = \begin{cases} 2^x & ,0 \le x < 1 \\ x & ,1 \le x \le 2 \\ x^2 - 4 & ,2 < x \le 3 \end{cases}$$

**(a)** Plot a graph showing $y = f(x)$ for $0 \le x \le 3$.

**(b)** Find the numerical value of each of the following limits:

**(I)** $\displaystyle\lim_{x \to 1^+} f(x).$ **(II)** $\displaystyle\lim_{x \to 1^-} f(x).$

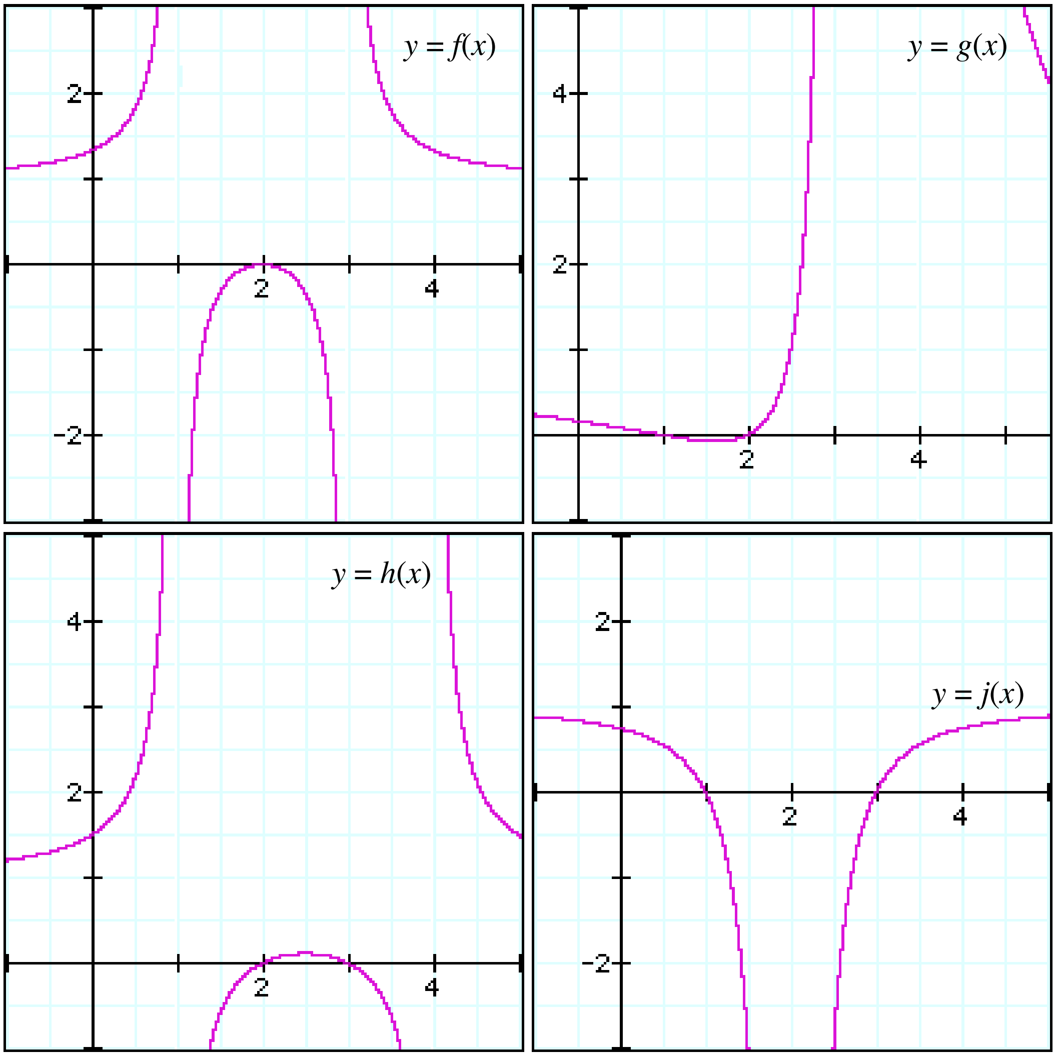
**(III)** $\displaystyle\lim_{x \to 2^+} f(x).$ **(IV)** $\displaystyle\lim_{x \to 2^-} f(x).$

**(c)** Often it is the case that the left hand limit or the right hand limit (sometimes both) is equal to the value of the function at a point. Does this always have to be the case? If you think so, explain why. If you don't think so, give an example to demonstrate your point of view.

**4.** In this problem, assume that the functions $f$, $g$, $h$ and $j$ are all rational functions with equations of the form:

$$y = \frac{(x - A)\cdot(x - B)}{(x - C)\cdot(x - D)}$$

where $A$, $B$, $C$ and $D$ are all numbers. The graphs of the four functions are shown in the diagrams below.




Each of the statements below shows how the numbers $A$, $B$, $C$ and $D$ are related to each other. Decide which statement corresponds to which graph.

**(a)** $A < C$, $C = D$, $D < B$.

**(b)** $A < B < C < D$.

**(c)** $C < B < A < D$.

**(d)** $C < A$, $A = B$, $B < D$.

**(e)** $A < C < B < D$.

**5.** Hungary is one of the few countries in the world where the population is decreasing. In 1990 the population of Hungary was approximately 10.8 million. Since 1990, the population has been decreasing by approximately 0.2% every year.

**(a)** Find an equation that gives the population of Hungary as a function of time.

**(b)** Calculate the derivative of the function that you found in Part (a). What are the units of your derivative?

**(c)** At what rate was the population of Hungary changing at the beginning of the year 2000? Give a practical interpretation of this number that could be understood by a person who was not familiar with calculus.

**(d)** Find an equation for the inverse of the function from Part (a).

**(e)** Calculate the derivative of the inverse that you found in Part (d).

**(f)** Evaluate the derivative of the inverse when the population of Hungary is equal to the population of New Zealand (3.4 million).

**(g)** Give a practical interpretation of the numerical value that you obtained in Part (f).

**6.** In this problem the function *f* will always refer to the function defined by the following equation:

$$f(x) = \sqrt{x}\,.$$

**(a)** Use the power rule (or any other short-cut rules that you know) to find an equation for the derivative $f'(x)$.

**(b)** Set up the difference quotient that you would use to find an equation for $f'(x)$.

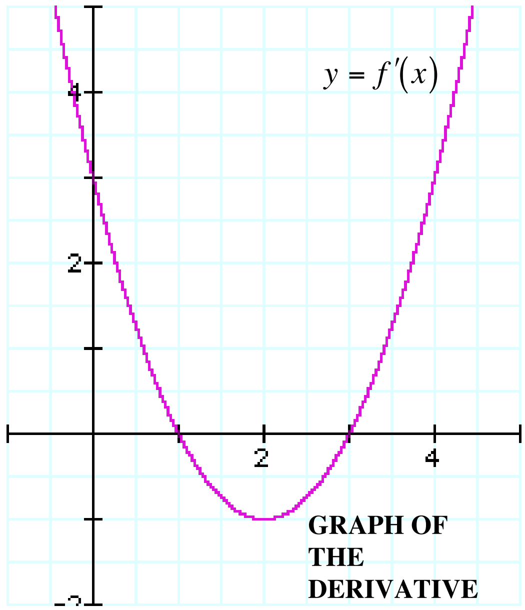
**(c)** Simplify your answer to Part (b) as much as possible. When you have finished simplifying, you should no longer have any factors of *h* in the denominator of your difference quotient.

**Note:** One identity that you might find helpful in Part (c) is:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\cdot\left(\sqrt{x+h}+\sqrt{x}\right)}{h\cdot\left(\sqrt{x+h}+\sqrt{x}\right)}.$$

**(d)** Use your answer to Part (c) to calculate an equation for the derivative $f'(x)$ using the limit definition of the derivative.

**7.** The graph given below is the graph of the **derivative** of a function *f*(*x*). In this problem the function *f*(*x*) will always refer to a function that has a derivative identical to the one shown in the graph below.




**(a)** Find the $x$-coordinates of all of the critical points of the function $f(x)$.

**(b)** Classify each of the critical points that you found as a minimum or a maximum. In each case, be careful to explain what information you are using from the derivative graph to make your decision.

**(c)** Find the $x$-coordinates of any points where the **concavity** of the original function $f(x)$ changes. Briefly explain how you know that the concavity changes at the point(s) you have located.

**(d)** Suppose the one other thing that you know about the function $f(x)$ is that $f(0) = 1$. Sketch a possible graph of $y = f(x)$ that is consistent with the derivative graph. Be careful to indicate the locations of all of the points that you found in Parts (a) and (c).

**8.** The golden-mantled ground squirrel (*Spermophilus saturatus* - see Figure 1[1]) is a small mammal that is common in many parts of North America, especially the Pacific north-west. These animals live in underground burrows. Typically, a ground squirrel spends about 16.5 hours underground and about 7.5 hours on the surface. The squirrels only forage for food when they are on the surface.

The diet of the golden-mantled ground squirrel includes leaves, bark, berries and seeds. Two of the most important food items are seeds (normally from pine and fir trees) and fungi (principally *Elaphomyces granulatus* - see Figure 2[2]).

Figure 1: The golden-mantled ground squirrel ( *Spermophilus saturatus*)

[1] Image source: http://www.washington.edu/burkemuseum/

[2] Image source: http://www.publish.csiro.au/ecos/Ecos91/Ecos91B.htm

The foraging habits and energy requirements of this squirrel have been studied to some extent[3]. The two scientific papers that this review problem are based on are:

Figure 2: Ground squirrel eating a piece of the fungus *Elaphomyces granulatus* .

- S.J. Cork and G.J. Kenagy. "Nutritional value of a hypogeous fungus for a forest-dwelling ground squirrel." *Ecology*, **70**(3): 577-586, 1989.

- G. J. Kenagy and D.F. Hoyt. "Speed and time-energy budget for locomotion in golden-mantled ground squirrels." *Ecology*, **70**(6): 1834-1839, 1989.

These scientists found that the amount of energy expended by a squirrel depended on whether it was foraging or resting. The average amount of energy that ground squirrels expend[4] (each hour) while either foraging or resting is shown in Table 1 below.

| Activity | Energy expenditure (kJ per hour) |
|---|---|
| Foraging | 13.695 |
| Resting | 2.73 |

Table 1: Average energy expenditures of ground squirrels

Two of the most important food sources for golden-mantled ground squirrels are seeds (from pine or fir) and fungus. The nutritional information, as well as the average amount of time required to find and consume these foods are shown in Table 2 below.

| Food | Energy (kJ) | % of energy digested | Search time (minutes) | Consumption time (minutes) |
|---|---|---|---|---|
| Seeds | 26.05 | 96 | 18 | 15 |
| Fungus | 17.3 | 52.2 | 10 | 3 |

Table 2: Nutritional and foraging information for ground squirrel common foods

**(a)** Suppose that $T$ represents the number of hours that a ground squirrel spends foraging each day. Find an equation that gives the energy expended by a squirrel as a function of $T$. What is the problem domain (i.e. set of $T$-values that make sense in this particular context) of your function?

**(b)** As in Part (a), let $T$ represent the number of hours that a ground squirrel spends foraging each day. Find an equation that represents the energy gained by a squirrel that forages in a home range where fungus is the main food, and an equation that represents the energy gained by a squirrel that forages in a home range where seeds[5] are the main source of food.

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[3] The golden-mantled ground squirrel enters a state of near hibernation during the winter, where it dramatically lowers its metabolism and lives on stored body fat. Exactly how the squirrel does this is of interest to researchers who hope to be able to find a way for humans to enter a similar state of "suspended animation." Two applications of this technology would include "freezing" astronauts for very long-duration space missions (e.g. to Mars) and temporarily suspending the death of critically injured people until they could be moved to suitable medical facilities.

[4] The numbers given here apply to an animal with a body mass of 150g, which is average for *Spermophilus saturatus*.

[5] Note the length of time that squirrels must spend consuming seeds. This is because the seeds of pine and fir must first be extracting from cones. This operation takes the squirrels a fair amount of time.

**(c)** Depending on the time of the year, the foraging habits of the squirrel can change. During Summer, the squirrels usually only forage for long enough to satisfy their immediate energy needs. However, in the Fall and Spring, the squirrels try to gain as much energy as they possibly can. Calculate the number of hours that a squirrel would be expected to spend foraging in each of the four cases listed below. In each case, show your calculation.

**Case 1:** It is Summer and the squirrel's home range offers mainly fungus to eat.

**Case 2:** It is Fall and the squirrel's home range offers mainly fungus to eat. (Remember the problem domain.)

**Case 3:** It is Summer and the squirrel's home range offers mainly seeds to eat.

**Case 4:** It is Fall and the squirrel's home range offers mainly seeds to eat. (Remember the problem domain.)

**(d)** Select one type of squirrel diet (fungus OR seeds). Using one set of axes, draw graphs showing:

- the energy expended by the squirrel as a function of the number of hours spent foraging

- the energy gained by the squirrel as a function of the number of hours spent foraging.

On your graphs, mark the locations of the two points that you calculated in Part (c). Based on what you have seen in other optimization problems and Part (d) of this problem, complete the following phrase:

"If $x$ is restricted to an interval $[a, b]$, then the maximum and minimum values of a function $f(x)$ occur either:

- at a point where $f'(x) = 0$, or,
- at a point where $f'(x)$ is difficult to define, or,
- at ______________________________________________ .

**9.** In this problem, $f(x)$ and $g(x)$ are functions that have derivatives. All that you can assume about them is

- $f'(2) = 7$
- $f(2) = 2$
- $g'(2) = -4$
- $g(2) = 18$.

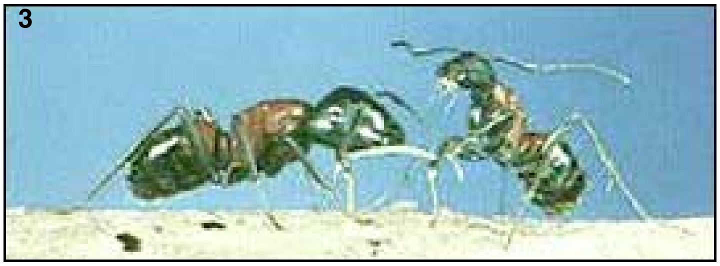
Use the information given about $f(x)$ and $g(x)$ to calculate the derivatives of the following functions.

**(a)** $h'(2)$, where $h(x) = f(x) \cdot g(x)$.

**(b)** $k'(2)$, where $k(x) = \dfrac{f(x)}{g(x)}$.

**(c)** $j'(2)$, where $j(x) = [f(x) + g(x)]^2$.

**10.** In the Costa Rican rainforest some species of ants (*Pseudomermex spinicola*, *Pseudomermex ferruginea* and *Pseudomermex nigrocinta* - see Figure 3[6]) live in acacia trees ( *Acacia collinsi*). The tree provides the ants with shelter and the ants work to eliminate competition from other plants. One of the things that the ants do is to clear away plants from around the acacia tree, leaving a circular patch of bare earth around the tree (see Figure 4[7]). According to Janzen (1966) when the colony is sufficiently large (more than 1200 ants) ants clear the ground 24 hours per day.



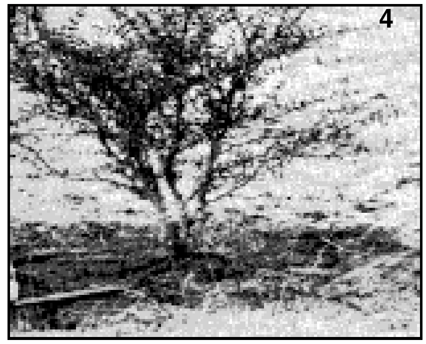




Figure 3: A pair of acacia ants (*Pseudomermex ferruginea* )

Figure 4: An acacia plant. The dark patch underneath the plant is the circle of bare earth that has been cleared by the ants.

Figure 5 (below) shows the ***INSTANTANEOUS RATE*** at which a large colony of ants clears the ground beneath an acacia tree. (The units of the instantaneous rate are square centimeters per hour.)

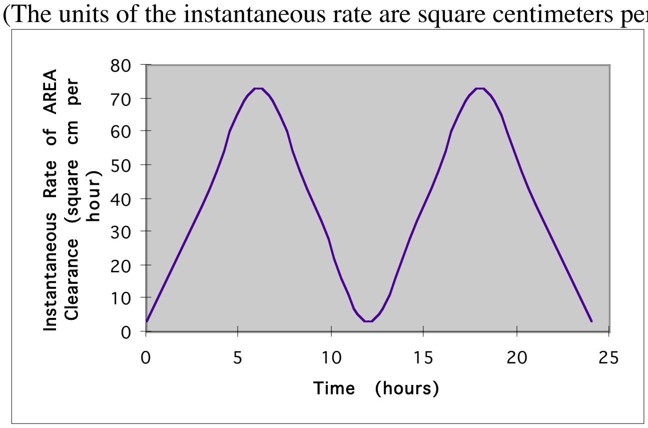



Figure 5: Graph showing instantaneous rate of change of AREA cleared.

**(a)** Suppose that $A(t)$ represents the amount of area (in square centimeters) cleared by the ants after they have been working for $t$ hours. Let $r(t)$ represent the radius of the circle (in centimeters) that the ants have cleared after they have been working for $t$ hours. Then $A(t)$ and $r(t)$ are related by the equation:

$$A(t) = \pi \cdot \left[r(t)\right]^2 = \pi \cdot r(t) \cdot r(t).$$

Find an equation that relates the derivative of the area function, $A'(t)$, to the radius function, $r(t)$, and the derivative of the radius function, $r'(t)$. Show details of your calculation.

**(b)** At 9am ($t = 9$) the radius of the circular area cleared by the ants had reached 50cm. What is the instantaneous rate of change of the *radius* of the circular area at 9am? Show details of your calculation.

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[6] Image source: http://www.abc.net.au/science/news/stories/s58491.htm

[7] Source: Janzen, D. H. (1966) "Coevolution of mutualism between ants and acacias in Central America." *Evolution*, **20**(3): 249-275.