Solutions: Test #2, Set #2

These answers are provided to give you something to check your answers against and to give you some idea of how the problems were solved. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.

1.(a) The limit is equal to 5.

1.(b) The limit does not exist as the height of the graph $y = f(x)$ does not settle down to any consistent value. Instead, it just oscillates back and forth.

1.(c) The limit does not exist as the left and right hand limits are not equal.

1.(d) The limit does not exist as the left and right hand limits are not equal.

1.(e) The limit is equal to 1.

1.(f) The limit does not exist as the left and right hand limits are not equal.

1.(g) The limit is equal to 1.

1.(h) The limit is equal to 2.

1.(I) The limit is equal to $+\infty$.

2.(a) The average speed is: $\dfrac{s(10)-s(0)}{10-0} = 26.4$ feet per second.

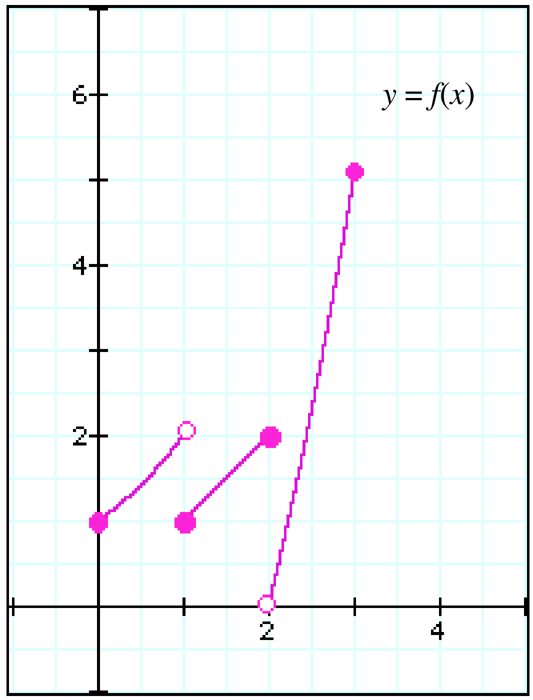
2.(b) The speedometer reading will be equal to $s'(10) = 52.8$ feet per second. There are 5280 feet in a mile and 3600 seconds in an hour, so the speedometer will read 52.8*3600/5280 ≈ 36 miles per hour.

2.(c) The speedometer reading will be equal to $s'(30) = 105.6$ feet per second. There are 5280 feet in a mile and 3600 seconds in an hour, so the speedometer will read 105.6*3600/5280 ≈ 72 miles per hour.

2.(d) The average speed over the interval $[10, 10 + k]$ will be: $\dfrac{2.64\cdot(10+k)^2 - 2.64\cdot 10^2}{k}$.

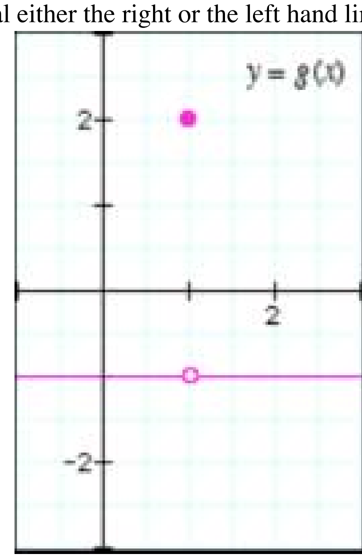
2.(e) The limit is equal to 52.8 (which also happens to be the derivative of $s(t)$ when $t = 10$).

3.(a) The graph of $y = f(x)$ is shown in the diagram below.



3.(b) **(I)** 1 **(II)** 2 **(III)** 0 **(IV)** 2.

3.(c) No it is not always the case. The diagram below shows the graph of a function g. The value of $g(1)$ is 2, but the left and right hand limits of g as x approaches 1 are both equal to -1. Therefore, the value of the function does not have to equal either the right or the left hand limit.



4.(a) $y = j(x)$.

4.(b) $y = g(x)$.

4.(c) $y = h(x)$.

4.(d) $y = f(x)$.

4.(e) This statement does not match any of the functions.

5.(a) Let T = number of years since 1990, and $P(T)$ = population of Hungary in millions of people. Then the equation for the Hungarian population is: $P(T) = 10.8 \cdot (0.998)^T$.

5.(b) The derivative of $P(T)$ is given by: $P'(T) = 10.8 \cdot (0.998)^T \cdot \ln(0.998)$. Evaluating this when $T = 10$ gives: $P'(10) = -0.021193$. The units of $P'(T)$ are millions of people per year.

5.(c) A practical interpretation of $P'(10) = -0.021193$ is that between $T = 10$ (i.e. the beginning of the year 2000) and $T = 11$ (i.e. the beginning of the year 2001) the population of Hungary would decrease by approximately 21193 people.

5.(d) The inverse of $P(T)$ is: $T(P) = \dfrac{\ln(P) - \ln(10.8)}{\ln(0.998)}$. (Note that you could also have used the common logarithm function here - in which case your equation would involve "log" rather than "ln.")

5.(e) The derivative of the inverse is given by: $T'(P) = \dfrac{1}{\ln(0.998) \cdot P}$.

5.(f) When $P = 3.4$, the derivative is: $T'(3.4) = -146.9$.

5.(g) One practical interpretation of this number is that it will take approximately 146.9 years for the population of Hungary to drop from 3.4 to 2.4 million.

6.(a) $f'(x) = (1/2)x^{-1/2}$.

6.(b) The difference quotient for $f(x)$ will be:

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

6.(c) Simplifying the difference quotient as much as possible gives:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x+h} - \sqrt{x}\right) \cdot \left(\sqrt{x+h} + \sqrt{x}\right)}{h \cdot \left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{x + h - x}{h \cdot \left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

6.(d) Taking the limit of the simplified difference quotient as $h \to 0$ gives: $f'(x) = \dfrac{1}{\sqrt{x} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$.

7.(a) The critical points of the original function, $f(x)$, are the points where the first derivative, $f'(x)$ is equal to zero. The critical points of $f(x)$ are located at the same x-coordinates as the x-intercepts of the derivative, $f'(x)$. If you inspect the graph, then you can see that these occur at $x = 1$ and $x = 3$.

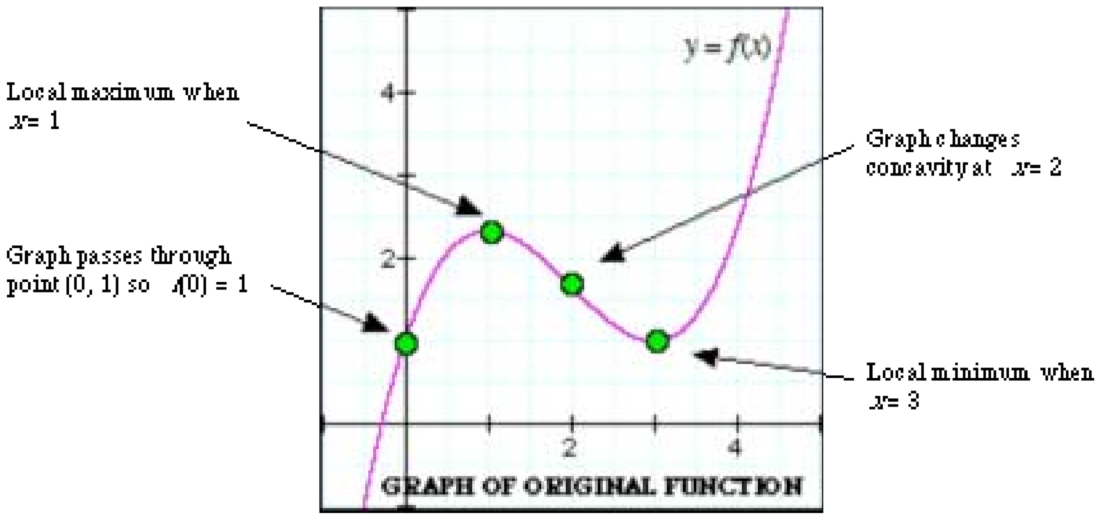
7.(b) **Classifying the critical points of $f(x)$:** If you look at the graph of the derivative, you can see that slightly to the left of $x = 1$, the derivative is positive and slightly to the right of $x = 1$, the derivative is negative. When the first derivative goes from positive to negative, the critical point is a local maximum.

Similarly, from the graph of the derivative, if you look slightly to the left of $x = 3$ then you can see that the derivative is negative whereas slightly to the right of $x = 3$ the derivative is positive. When the first derivative goes from negative to positive the critical point is a local minimum.

7.(c) The definition of a point of inflection is that this is where the graph of $f(x)$ changes concavity. You can locate this point from the derivative graph by following the following reasoning:

When the derivative is decreasing, the graph of the original function $f(x)$ is concave down. When the derivative is increasing, the graph of the original function $f(x)$ is concave up. The point(s) of inflection of $f(x)$ will occur at any point where the derivative graph goes from increasing to decreasing or vice versa. If you look at the derivative graph, you will be able to see that the only point where this occurs is at $x = 2$.

7.(d) A possible[1] graph of the original function $y = f(x)$ is shown below.



8.(a) There are two immediate interpretations of what the question is asking here. However, only one interpretation will be suitable for use later in the problem.

The first interpretation is that the question is asking you to find a function for the amount of energy that a squirrel expends while actually foraging. As we are told that a 150g squirrel expends 13.695 kJ per hour while foraging, this would suggest a function of the form:

$$C_{first}(T) = 13.695 \cdot T.$$

Where this interpretation is not so useful is that it does not take into account that a 150g squirrel expends 2.73 kJ of energy per hour even when resting. This suggests a second interpretation for the function - the amount of energy that the squirrel expends foraging plus the amount of energy that the squirrel expends while resting. If the squirrel forages for T hours then it will rest for $24 - T$ hours before foraging again. The total amount of energy expended by the squirrel is then:

$$C(T) = 13.695 \cdot T + 2.73 \cdot (24 - T) = 65.52 + 10.965 \cdot T.$$

This second interpretation is the one that will make sense when used later in the problem.

[1] There are many possible answers here, although correct answers will quite closely resemble the graph given. What we were looking for was that you included the point (0, 1) on the graph, had the critical points and inflection point in the right places, indicated the locations of these special points and had a plausible shape for the graph overall.

The problem domain will be the set of possible values of T that make sense as potential squirrel foraging times. As squirrels typically only spend 7.5 hours above ground each day, and only forage when above ground, the appropriate problem domain would be:

$$0 \leq T \leq 7.5.$$

8.(b) **Energy Gained by a Squirrel Eating Fungus:** Each piece of fungus takes the squirrel a total of 13 minutes to locate and consume (10 to locate, 3 to consume). So in one hour, a squirrel can locate and consume:

$$\frac{60}{13} = 4.6154 \text{ fungi.}$$

Each piece of fungus yields 17.3 kJ, of which the squirrel is able to extract 52.2%. In energy terms, each hour the squirrel is able to obtain:

$$\frac{60}{13} * 17.3 * 0.522 = 41.6798 \text{ kJ.}$$

Therefore, if the squirrel forages for a total of T hours, then the squirrel will gain:

$$R_{fungus}(T) = 41.6798 * T \text{ kJ.}$$

Energy Gained by a Squirrel Eating Seeds:

Each cone of seeds (pine or fir) takes the squirrel a total of 33 minutes to locate and consume (18 to locate, 15 to consume). So in one hour, a squirrel can locate and consume:

$$\frac{60}{33} = 1.818 \text{ fungi.}$$

Each cone of seeds yields 26.05 kJ, of which the squirrel is able to extract 96%. In energy terms, each hour the squirrel is able to obtain:

$$\frac{60}{33} * 26.05 * 0.96 = 45.4691 \text{ kJ.}$$

Therefore, if the squirrel forages for a total of T hours, then the squirrel will gain:

$$R_{seeds}(T) = 45.4691 * T \text{ kJ.}$$

8.(c) The calculations and conclusions for each individual case are presented one-by-one below. The results of these calculations are summarized in the following table.

Case	Amount of time squirrel should spend foraging (hours)
1	2.13
2	7.5
3	1.899
4	7.5

Case 1: It is Summer and the squirrel feeds mainly on fungus.

The squirrel needs to consume enough kJ of energy to offset the amount of energy that it expends each day. That is, the time that the squirrel should spend foraging is the value of *T* that solves the equation:

$$R_{fungus}(T) = C(T).$$

Solving for *T*:

$$41.6798 * T = 65.52 + 10.965 * T$$

$$T = \frac{65.52}{30.7148} = 2.13 \text{ hours.}$$

Case 2: It is Fall and the squirrel feeds mainly on fungus.

Here the squirrel needs to maximize the net amount of energy that it gains. In financial terms, if the energy that the squirrel gains from foraging is likened to revenue and the energy that the squirrel expends is likened to costs, then the squirrel wants to maximize profits. Writing $P_{fungus}(T)$ for the net energy gained by the squirrel each day:

$$P_{fungus}(T) = R_{fungus}(T) - C(T) = 30.7148 * T - 65.52.$$

The squirrel wants to make this as large as possible. However, if you take the derivative of $P_{fungus}(T)$ then you get 30.7148 which cannot possibly be set equal to zero. Another approach is needed.

Graphing $y = P_{fungus}(T)$ for the problem domain of $0 \le T \le 7.5$ gives the graph shown in Figure 4 below.

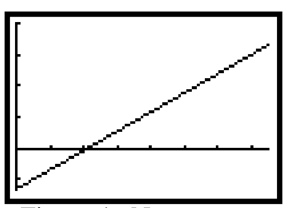


Figure 4: Net energy graph for a squirrel consuming fungus.

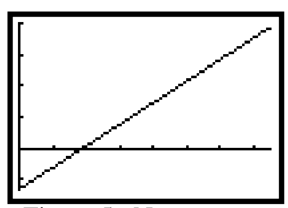


Figure 5: Net energy graph for a squirrel consuming seeds.

Figure 4 shows that $P_{fungus}(T)$ is maximized when *T* is allowed to be as large as possible - in this case, $T = 7.5$. Hence, the squirrel should spend 7.5 hours foraging each day.

Case 3: It is Summer and the squirrel feeds mainly on seeds.

The squirrel needs to consume enough kJ of energy to offset the amount of energy that it expends each day. That is, the time that the squirrel should spend foraging is the value of *T* that solves the equation:

$$R_{seeds}(T) = C(T).$$

Solving for *T*:

$$45.4691 * T = 65.52 + 10.965 * T$$

$$T = \frac{65.52}{34.5041} = 1.899 \text{ hours.}$$

Case 4: It is Fall and the squirrel feeds mainly on seeds.

Here the squirrel needs to maximize the net amount of energy that it gains. In financial terms, if the energy that the squirrel gains from foraging is likened to revenue and the energy that the squirrel expends is likened to costs, then the squirrel wants to maximize profits. Writing $P_{seeds}(T)$ for the net energy gained by the squirrel each day:

$$P_{seeds}(T) = R_{seeds}(T) - C(T) = 34.5041 * T - 65.52.$$

The squirrel wants to make this as large as possible. However, if you take the derivative of $P_{seeds}(T)$ then you get 34.5041 which cannot possibly be set equal to zero. Another approach is needed.

Graphing $y = P_{seedss}(T)$ for the problem domain of $0 \leq T \leq 7.5$ gives the graph shown in Figure 5. Figure 5 shows that $P_{seeds}(T)$ is maximized when T is allowed to be as large as possible - in this case, $T = 7.5$. Hence, the squirrel should spend 7.5 hours foraging each day.

8.(d) The graphs for squirrels on a diet consisting mainly of fungus are shown below.

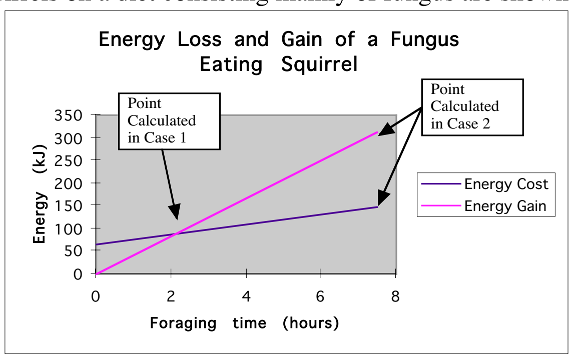


Figure 6: Fungus eating squirrel

The graphs for squirrels on a diet consisting mainly of seeds are shown below.

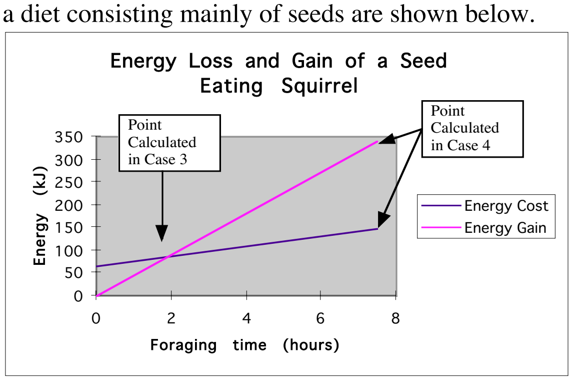


Figure 7: Seed eating squirrel

From these graphs it seems to be the case that:

> "If x is restricted to an interval $[a, b]$, then the maximum and minimum values of a function $f(x)$ occur either:
>
> - at a point where $f'(x) = 0$, or,
> - at a point where $f'(x)$ is difficult to define, or,
> - at one of the endpoints ($x = a$ or $x = b$) of the interval $[a, b]$."

9.(a) The function $h(x)$ is defined to be the *product* of $f(x)$ and $g(x)$, so to find the derivative of $h(x)$ you should use the product rule:

$$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

So, when $x = 2$:

$$h'(2) = f'(2)\cdot g(2) + f(2)\cdot g'(2) = 7*18 + 2*(-4) = 118.$$

9.(b) The function $k(x)$ is defined to be the *quotient* of $f(x)$ and $g(x)$, so to find the derivative of $k(x)$ you should use the quotient rule:

$$k'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}.$$

So, when $x = 2$:

$$k'(2) = \frac{f'(2)\cdot g(2) - f(2)\cdot g'(2)}{[g(2)]^2} = \frac{7*18 - 2*(-4)}{18^2} = 0.41358.$$

9.(c) The function $j(x)$ can be simplified by multiplying out the brackets before differentiating:

$$j(x) = f(x)\cdot f(x) + 2\cdot f(x)\cdot g(x) + g(x)\cdot g(x).$$

Each of the individual terms in this expanded version of $j(x)$ can be differentiated using the *product rule*, as each is a *product* of two functions. Therefore:

$$j'(x) = f'(x)\cdot f(x) + f(x)\cdot f'(x) + 2\cdot f'(x)\cdot g(x) + 2\cdot f(x)\cdot g'(x) + g'(x)\cdot g(x) + g(x)\cdot g'(x)$$

which can be factored to give:

$$j'(x) = 2\cdot[f(x) + g(x)]\cdot[f'(x) + g'(x)].$$

So, when $x = 2$:

$$.\ j'(2) = 2\cdot[f(2) + g(2)]\cdot[f'(2) + g'(2)] = 120.$$

10.(a) The key relationship here is the relationship between the area function, $A(t)$, and the radius function, $r(t)$:

$$A(t) = \pi\cdot r(t)\cdot r(t).$$

This is a *product* of two functions ($r(t)$ and $r(t)$) so when differentiating it is appropriate to use the *product rule*.

$$A'(t) = \pi\cdot r'(t)\cdot r(t) + \pi\cdot r(t)\cdot r'(t) = 2\pi\cdot r(t)\cdot r'(t).$$

This is the desired relationship between the derivative of the area function, $A'(t)$, the radius function, $r(t)$, and the derivative of the radius function, $r'(t)$.

10.(b) The quantity that you need to calculate here is: $r'(9)$. Rearranging the equation from Part (a) to make $r'(t)$ the subject gives the following:

$$r'(t) = \frac{A'(t)}{2\cdot\pi\cdot r(t)}.$$

We were told that when $t = 9$, the radius had reached 50 cm. So $r(9) = 50$. From the graph of the instantaneous rate of change of the area, you can read off the value for the derivative at $t = 9$:

$$A'(t) = 38.$$

Substituting these values into the expression for $r'(t)$ gives:

$$r'(9) = \frac{A'(9)}{2 \cdot \pi \cdot r(9)} = \frac{38}{100 \cdot \pi} = 0.121 \text{ cm per hour.}$$

So at 9am, the radius of the cleared area beneath the acacia tree is increasing at a rate of approximately 0.121 cm per hour.