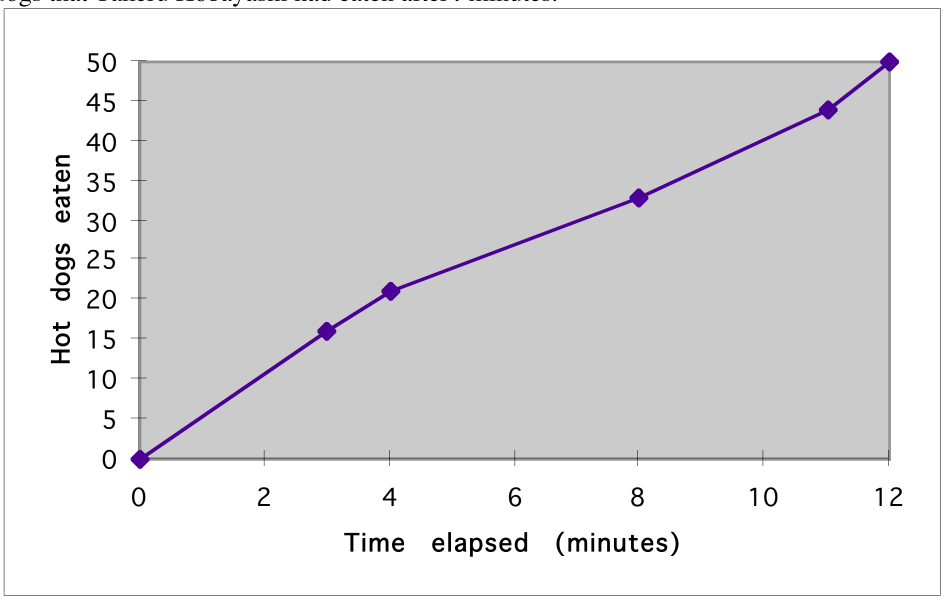
# Solutions: Test #2, Set #3

**These answers are provided to give you something to check your answers against and to give you some idea of how the problems were solved. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.**

**1.(a)** The graph shown below gives both the data points from the table given in the homework assignment as well as line segments joining these points. These line segments represent the collection of linear functions that you could use to create an equation for $N(t)$, the number of hot dogs that Takeru Kobayashi had eaten after $t$ minutes.




**1.(b)** Below, the calculations for the first two linear functions are shown in some detail.

**Calculation of First Linear Function**

The first linear function goes through the points (0, 0) and (3, 16). The slope of the linear function is the change in the number of hot dogs eaten divided by the change in time:

$$slope = \frac{\Delta hot\_dogs}{\Delta time} = \frac{16-0}{3-0} = \frac{16}{3}.$$

The intercept can be calculated by substituting the slope of $m = 16/3$ and the coordinates of either point into the equation for a linear function,

$$y = m \cdot x + b.$$

Doing this gives:

$$0 = (16/3)*0 + b$$

so that $b = 0$ and the equation for the linear function that represents the first portion of the graph shown in Part (a) is:

$$y = \tfrac{16}{3}x$$

where $x$ is the time elapsed in minutes and $y$ is the total number of hot dogs eaten by Takeru Kobayashi.

**Calculation of Second Linear Function**

The first linear function goes through the points (3, 16) and (4, 21). The slope of the linear function is the change in the number of hot dogs eaten divided by the change in time:

$$slope = \frac{\Delta hot\_dogs}{\Delta time} = \frac{21-16}{4-3} = \frac{5}{1} = 5.$$

The intercept can be calculated by substituting the slope of $m = 5$ and the coordinates of either point into the equation for a linear function,

$$y = m \cdot x + b.$$

Doing this gives:

$$16 = (5)*3 + b$$

so that $b = 1$ and the equation for the linear function that represents the second portion of the graph shown in Part (a) is:

$$y = 5x + 1$$

where $x$ is the time elapsed in minutes and $y$ is the total number of hot dogs eaten by Takeru Kobayashi.

**The Collection of all of the Linear Functions for *N*(*t*)**

The functions, along with the intervals that each equation applies to, are shown in Table 1 below.

| Equation for linear function | Interval on which this equation applies |
|---|---|
| $y = \frac{16}{3}t$ | $0 \le t < 3$ |
| $y = 5t + 1$ | $3 \le t < 4$ |
| $y = 3t + 9$ | $4 \le t < 8$ |
| $y = \frac{11}{3}t + \frac{11}{3}$ | $8 \le t < 11$ |
| $y = 6t - 22$ | $11 \le t \le 12$ |

Table 1

Using the notation that is conventional for functions that are defined in a piecewise fashion, the function $N$ could be defined as:

$$N(t) = \begin{cases} y = \frac{16}{3}t & , 0 \le t < 3 \\ y = 5t + 1 & , 3 \le t < 4 \\ y = 3t + 9 & , 4 \le t < 8 \\ y = \frac{11}{3}t + \frac{11}{3} & , 8 \le t < 11 \\ y = 6t - 22 & , 11 \le t \le 12 \end{cases}$$

**1.(c)** By looking at the graph in Part (a), you can see that Takeru Kobayashi reached the former world record of 28.125 hot dogs somewhere between $t$ = 4 and $t$ = 8 minutes into the competition. During this time period, the relevant equation is:

$$y = 3t + 9.$$

Setting $y$ = 28.125 and solving for $t$ gives:

$$t = \frac{28.125 - 9}{3} = 6.375.$$

In other words, after about 6.375 minutes of the competition had elapsed, Takeru Kobayashi had eaten 28.125 hot dogs.

**1.(d)** The answer here depends on what you think constitutes eating a hot dog. Do you just have to swallow the hot dog (in which case the function must always increase since the number of hot dogs that have been swallowed can never go down) or does it mean that you have to both swallow the hot dog and retain it in your stomach? The official rules of the Nathan's Famous Hot Dog Eating Contest stipulate that it is the second – you must not only eat the hot dog, you must also keep it in your stomach. In this case, if a competitor were to regurgitate, the function could go down, as the hot dogs that were brought up would be deducted from his or her[1] total.

**2.(a)** The graph of $d(t)$ is shown in the diagram on the next page.

**2.(b)** Factoring the equation for $d(t)$ gives: $d(t) = -16t(t - 3)$. This equation is equal to zero when $t$ = 0 and when $t$ = 3. The first time that $d(t)$ = 0 is when $t$ = 0. This is probably when the tomato was launched into the air. The second time when $d(t)$ = 0 is when $t$ = 3. This is probably when the tomato landed on the ground.

**2.(c)** The time when height, $d(t)$, was maximum can be found by locating the time when $d'(t)$ = 0. Differentiating the equation for $d(t)$ gives:

$$d'(t) = -32t + 48.$$

Setting this derivative equal to zero and solving for $t$ gives that the tomato reaches its maximum height when $t$ = 1.5.

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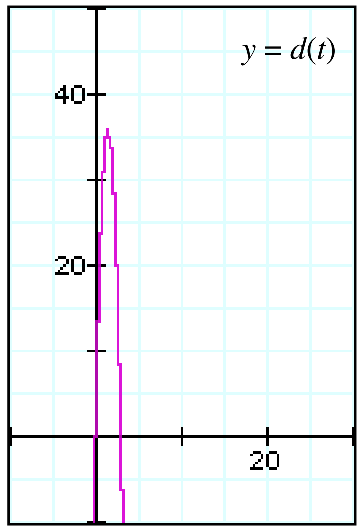
[1] Believe it or not women compete in this event. Last year, a Japanese woman weighing about 110 pounds before the competition (Ms. Hirofumi Nakajima) came in third behind Kobayashi and Arai, beating out all of the American contestants.




**2.(d)** The value of the maximum height will be given by $d(1.5) = 36$ feet.

**2.(e)** The equation for $d(t)$ in vertex form is: $d(t) = -16(t - 1.5)^2 + 36$.

**2.(f)** The values of $t$ and $d(t)$ at maximum height are the $x$- and $y$-coordinates of the vertex of the graph shown in Part (a). If the coordinates of the vertex of a quadratic equation are $(h, k)$ then the equation for the quadratic in vertex form is: $q(x) = a\cdot(x - h)^2 + k$. So, since you know $h = 1.5$ and $k = 36$ from Parts (d) and (e), you could have obtained the vertex form by just plugging these numbers into $q(x) = a\cdot(x - h)^2 + k$. The value of $a$ could be determined by plugging one of the points from Part (b) into this equation and solving for $a$.

**3.(a)** Cup A - Graph 3. Cup B - Graph 4. Cup C - Graph 1. Cup D - Graph 2.

**3.(b)** $\left(d^{-1}\right)'(10) = 0.7$.

**4.(a)** As $k(x)$ is a product of two functions, you use the product rule to differentiate it. This gives:

$$k'(x) = \left(4x^3 - 1\right)\cdot\left(\tfrac{1}{2}x^2 + 2\right) + (x)\cdot\left(x^4 - x - 1\right).$$

**4.(b)** As $m(x)$ is a quotient of two functions, you use the quotient rule to differentiate it. Doing this gives $m'(2) = 6$.

**4.(c)** You have enough information to work out $m(2)$, as you are told that $h(2) = 1$ and you can use the equation for $g(x)$ to work out that $g(2) = 4$. Therefore, $m(2) = 4$. The derivative $m'(2)$ is approximately how much the function $m$ will change when $x$ is increased from $x = 2$ to $x = 3$. Since $m'(2) = 6$, the function will increase by approximately 6 when $x$ is increased from $x = 2$ to $x = 3$. Therefore, the value of $m(3) \approx 4 + 6 = 10$.

**5.(a)** $f'(x) = \frac{3}{2}x^2 - 2x$.

**5.(b)** When $x = 0$, the graph of $y = g'(x)$ has height 3. This means that the derivative of $g$ will be equal to three at that point. In symbols, that means that: $g'(0) = 3$. The derivative of $m(x)$ at $x = 0$ is: $m'(0) = f'(0) + g'(0) = 0 + 3 = 3$.

**5.(c)** $r(x) = m'(x) = f'(x) + g'(x) = \frac{3}{2}x^2 - 2x + 3.$

**6.(a)** The critical points of the function are the points where $f'(x) = 0$. The derivative of $f(x)$ is given by:

$$f'(x) = 3x^2 - 5.$$

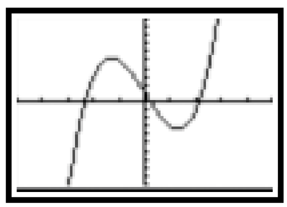
Setting this equation equal to zero and solving for $x$ gives: $x = \pm\sqrt{(5/3)}$. Evaluating the function $f(x)$ at these x-values gives the coordinates of the critical points:

$$(-(5/3)^{1/2}, 3.3) \text{ and } ((5/3)^{1/2}, -3.3).$$

**6.(b)** The function is increasing where the derivative is positive and decreasing where the derivative is negative. The function $f(x)$ is:

- Increasing on intervals: $-\infty < x < -(5/3)^{1/2}$ and $(5/3)^{1/2} < x < \infty$.
- Decreasing on interval: $-(5/3)^{1/2} < x < (5/3)^{1/2}$.

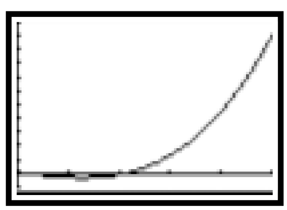
**6.(c)** Points of inflection usually occur where the concavity of the original function changes. It is always a good idea to check the graph of the function to make sure that there really is a point of inflection where you think there might be one. Checking the graph of $y = f(x)$ (see below) shows that the concavity of $f(x)$ appears to change at $x$=0, so there is a point of inflection at $x$=0.



**6.(d)** The function is concave up where the derivative is increasing and concave down where the derivative is decreasing. The function is then:

- Concave down on interval: $-\infty < x < 0$
- Concave up on interval: $0 < x < \infty$.

**6.(e)** The global minimum is the highest value that the function can have when $x$ is restricted to the values $0 \leq x \leq 5$. Inspecting the graph of $y = f(x)$ (see below) shows that the global maximum is: (5, 101).



**6.(f)** By inspecting the graph, the global minimum is located at the minimum that was detected in Part (a). Therefore, the global minimum is: $(-(5/3)^{1/2}, -3.3)$

**7.(a)** The units of the derivative will be barrels of oil per year.

**7.(b)** The sign of the derivative (when $t$ = 1999) will be negative. This is because more petroleum was consumed in 1999 than was produced, so the world petroleum resources were reduced.

**7.(c)** The statement means that between 1980 and 1981, the amount of petroleum in the world changed by approximately $R$ barrels. Since $R$ is negative, this means that the amount of petroleum in the world was about $|R|$ barrels less in 1981 as compared to 1980.

**7.(d)** The output of the Motunui plant will make the derivative slightly less negative. The output of the Motunui plant will not be sufficient to actually change the derivative from negative to positive, but it will make the derivative slightly less negative.

**8.(a)** Graph II. The biggest power of $x$ on the top is $x^2$ whereas the biggest power of $x$ on the bottom is $x^1$. As the power of $x$ on top is greater, there will be no horizontal asymptotes. Graph II is the only one that doesn't look as though it has any horizontal asymptotes.

**8.(b)** Graph V. The numerator of this rational function is zero at $x = +1$ and at $x = -1$. Therefore, the graph of this ration function will have $x$-intercepts at $x = +1$ and $x = -1$. Graph V is the only graph that shows $x$-intercepts in these locations.

**8.(c)** Graph IV. We are looking for a graph with $x$-intercepts at $x = +2$ and $x = -1$, and a vertical asymptote at $x = 0$. Both graphs I and IV show these features. To decide between then, the horizontal asymptote of the rational function should appear at $y = +3$. Therefore, graph I cannot be the right graph, as it shows a horizontal asymptote at $x = -3$.

**8.(d)** Graph III. Want a graph with just one $x$-intercept located at $x = +1$. Graph III is the only graph that shows this feature.

**9.(a)** The difference quotient is shown below. (I have simplified the difference quotient a little in anticipation of Part (b) of this problem.)

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{x-(x+h)}{h\cdot x\cdot(x+h)} = \frac{-1}{x\cdot(x+h)}$$

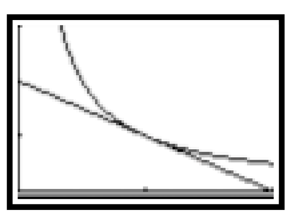
**9.(b)** The derivative is the limiting value of the difference quotient as $h \to 0$. Using the simplified difference quotient obtained in Part (a) above gives: $f'(x) = -1/x^2$.

**9.(c)** Re-writing the equation for $f(x)$ gives: $f(x) = x^{-1}$. Using the power rule for derivatives: $f'(x) = -x^{-2}$.

**9.(d)** Substituting $x = 1$ into the equation for the derivative gives the slope of the tangent line. So, the slope of the tangent line is -1. Substituting $m = -1$, $x = 1$ and $y = 1$ into the equation for a linear function gives the intercept, $b$, of the equation for the tangent line: $b = 2$. So, the equation for the tangent line is:

$$y = -x + 2.$$

**9.(e)** A plot showing the graph of $y = f(x)$ and the graph of $y = -x + 2$ is shown below.



**10.** Of the four available graphs, graph (d) is closest to satisfying the requirements for $f$. Neither (b) nor (c) have the derivative equal to zero when $x = 0$. Graph (a) is increasing when $x = -1$ and decreasing when $x = +1$, which is the precise opposite of what is wanted for $f$.