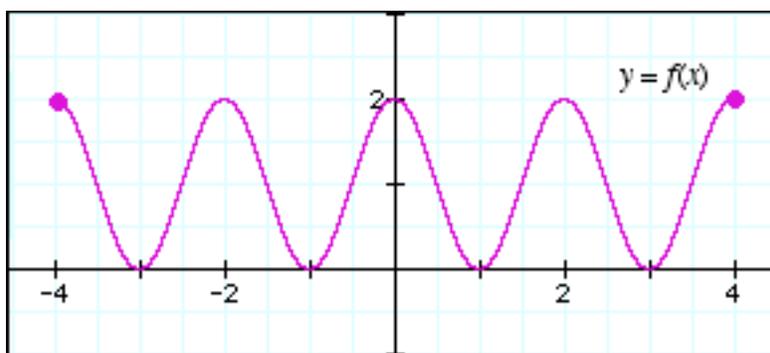


Practice Problems: Final Exam – Set #1

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.
 2. The test will include twelve problems (each with multiple parts).
 3. You will have 3 hours to complete the test.
 4. You may use your calculator and one page (8" by 11.5") of notes on the test.
 5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.
 6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.
 7. Good sources of help:
 - Section leaders' office hours (posted on Xa web site).
 - Math Question Center (during the reading period).
 - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.
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1. In this problem, the function $f(x)$ will always refer to the function defined by the graph given below. The domain of the function $f(x)$ is the interval $[-4, 4]$. Note that this interval does include the end-points $x = -4$ and $x = 4$.



In this problem, the function $g(x)$ will always refer to the function defined by the equation:

$$g(x) = [f(x)]^2 - 2 \cdot f(x).$$

- (a) What is the domain of the function $g(x)$?
- (b) Find the x -coordinates of the point(s) where the derivative of $f(x)$ is equal to zero.

- (c) Express the derivative of $g(x)$, $g'(x)$, in terms of $f(x)$ and the derivative of $f(x)$.
- (d) Find the x -coordinates of all points where the derivative of $g(x)$ is equal to zero.
- (e) For each of the points that you found in Part (d), determine whether the point is a local maximum, local minimum or neither.

2. Wild rabbits are the most serious animal pest on the island continent of Australia and the second most serious pest (after the marsupial possum) in the island nation of New Zealand. In both countries wild rabbits originally introduced from Europe have been responsible for extensive ecological damage¹. The annual cost of efforts to control the rabbit population and loss of agricultural productivity is estimated to be in the region of \$600 million per year².

Given that rabbits have created serious economic and ecological problems for Australia and New Zealand, it may come as a surprise that rabbits were intentionally released in both countries by European settlers in the 1800's. The introduction of rabbits to Australia is often traced to the importation and release of 24 rabbits by Thomas Austin in 1859³ on his property in southern Victoria. In 1866, there were at least 14,253 rabbits on Mr. Austin's property and by 1869 this population had grown to at least 2,033,000 rabbits. These numbers are summarized in the table below.

Year	Number of years since introduction	Number of rabbits
1859	0	24
1866	7	14,253
1869	10	2,033,000

- (a) Based on the figures given in the table above, is the number of rabbits a linear function of the number of years since introduction? Provide evidence for your answer.
- (b) Based on the figures given in the table, is the number of rabbits an exponential function of the number of years since introduction? Provide evidence for your answer.
- (c) Based on the figures given in the table, is the number of rabbits a power function of the number of years since introduction? Provide evidence for your answer.
- (d) Which of these three functions (linear, exponential or power) does the best job in representing the number of rabbits as a function of the number of years since introduction? Find an equation for this function.
- (e) Use your function to predict the number of rabbits that are currently living on Mr. Austin's property 143 years after they were first introduced.

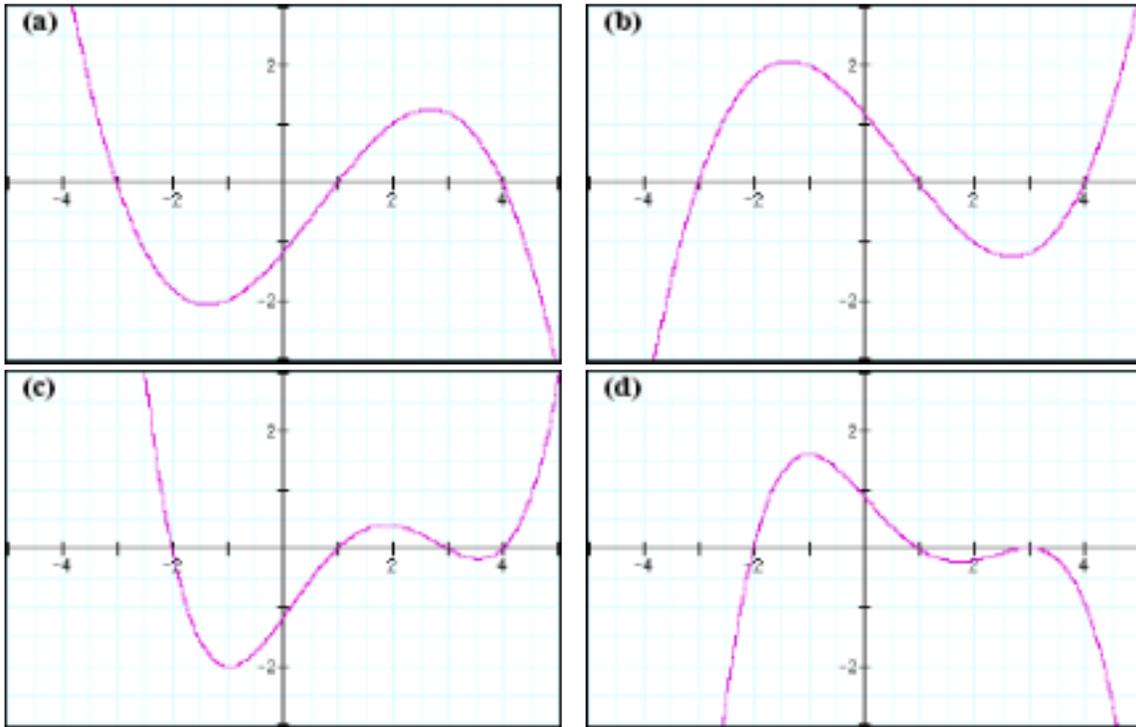
¹ Source: Brian D. Cooke. The effects of rabbit grazing on regeneration of sheoaks, *Allocasuarina verticillata*, and salt-water ti-trees, *Meloleuca halmaturorum*, in the Coorong National Park. *Australian Journal of Ecology*, **13**(1): 11-20, 1987.

² Source: CSIRO, Australia. Environmental damage by wild rabbits. *CSIRO Media Release*, August 17, 1996.

³ Source: E.C. Rolls. *They All Ran Wild*. Sydney, Australia: Angus and Robertson Publishers, Inc., 1969

- (f) A wild rabbit usually has a mass of a few kilograms. Do you think that the estimate from Part (e) is reasonable or not? For comparison, the mass of the sun is about 2×10^{30} kilograms. Sketch more realistic graph showing the number of rabbits as a function of time.

3. Find an equation for each of the polynomial functions shown below. Normally, your last step in such a calculation would be to determine the constant of proportionality, k , for each function. Instead of doing that here, simply determine whether the constant should be positive or negative.



4. In this problem the function $f(x)$ is a function whose derivative is always defined. The only two pieces of information that you can assume about $f(x)$ are given below:

- $f(1) = 1$

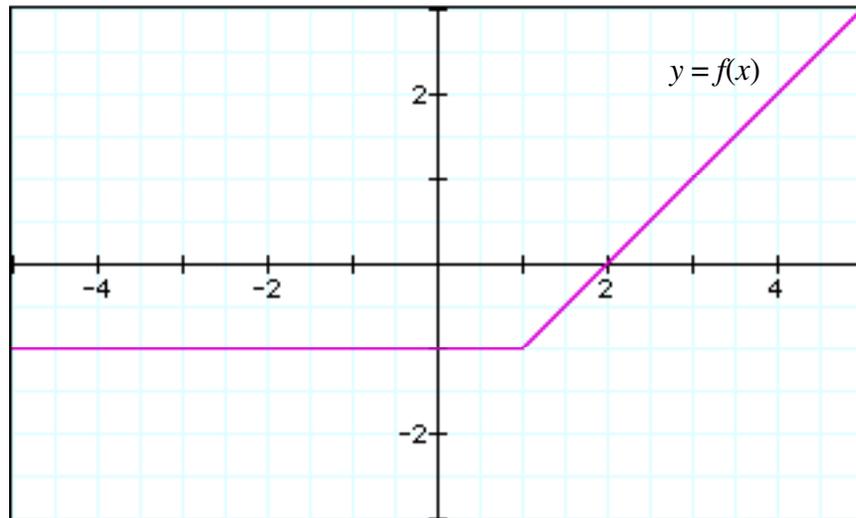
- $f'(1) = 1$

In this problem, the function $g(x)$ will always refer to the function defined by the equation given below:

$$g(x) = \frac{f(x)}{x}.$$

- (a) Set up the difference quotient that you need to evaluate in order to find $g'(1)$ using the limit definition of the derivative.
- (b) Simplify the difference quotient from Part (a) as much as you possibly can.
- (c) Take the limit as $h \rightarrow 0$ to calculate $g'(1)$.
- (d) Use the quotient rule to confirm your answer to Part (c).

5. In this problem, $f(x)$ will always refer to the function defined by the graph shown below.

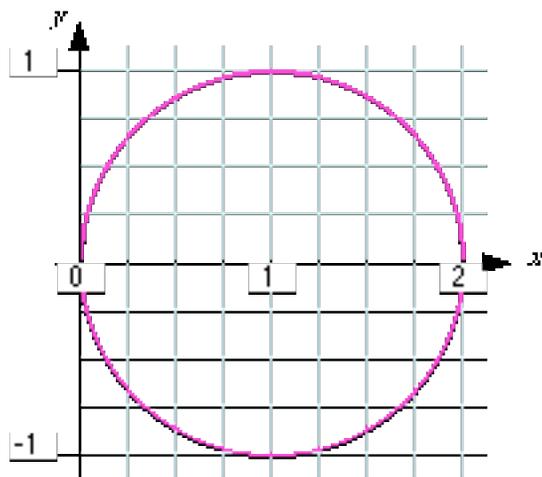


In this problem, the function $g(x)$ will always refer to the function defined by the equation:

$$g(x) = f(x) \cdot e^x.$$

- Write down an equation for the function $f(x)$. (Hint: don't try to write down a single formula. Instead break $f(x)$ up into pieces.)
- What is the domain of the derivative $g'(x)$?
- Write down an equation for the derivative $g'(x)$. (Hint: don't try to write down a single formula, instead break $f(x)$ up into pieces.)
- Write down an equation for the second derivative $g''(x)$. (Hint: don't try to write down a single formula. Instead break $f(x)$ up into pieces.)
- Find the intervals over which $g(x)$ is concave up and the intervals over which $g(x)$ is concave down.

6. The diagram given below shows a curve in the xy - plane.



- (a) The x - and y -coordinates of the points that lie on this curve satisfy the equation:

$$y^2 + (x - 1)^2 = 1.$$

Is it possible to find a function $f(x)$ so that the curve shown above is the graph of $y = f(x)$? If so, find an equation for $f(x)$. If not, explain why not.

- (b) Find an equation for the derivative of y with respect to x - that is, an equation for $\frac{dy}{dx}$.
- (c) The portion of the curve that lies above the x -axis can be represented by the equation:

$$y = \sqrt{1 - (x - 1)^2}.$$

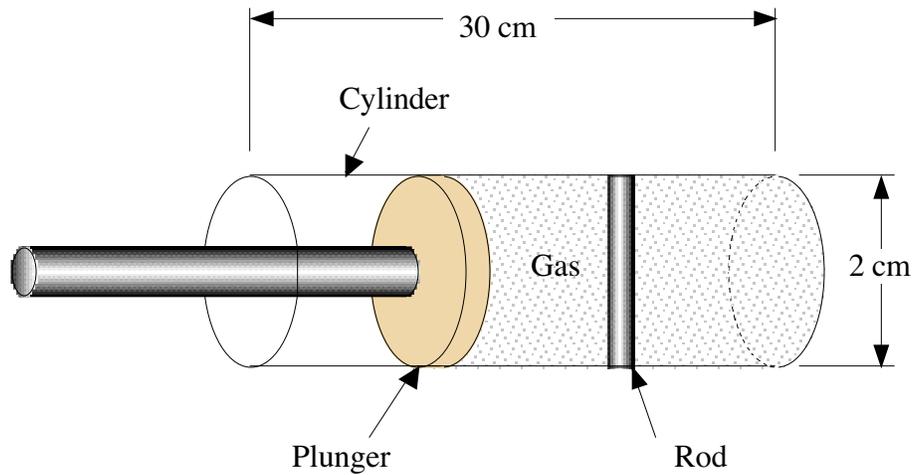
Use this explicit equation for y as a function of x to find an equation for $\frac{dy}{dx}$.

- (d) In Parts (b) and (c) of this problem, you found an equation for $\frac{dy}{dx}$ in two different ways. Chances are, the two equations that you found for $\frac{dy}{dx}$ look quite different as well. Are these two equations really the same or are they different in some important way?

7. During an experiment in an introductory chemistry class, a gas undergoes adiabatic compression (meaning that no heat is added or removed) in a glass tube 30 cm long and 2 cm in diameter. As the gas is compressed, a student records the following data for pressure and volume:

Pressure (atmospheres)	1.0	1.1	1.3	1.5	2.0	2.5
Volume (cm ³)	94	86	80	71	58	49

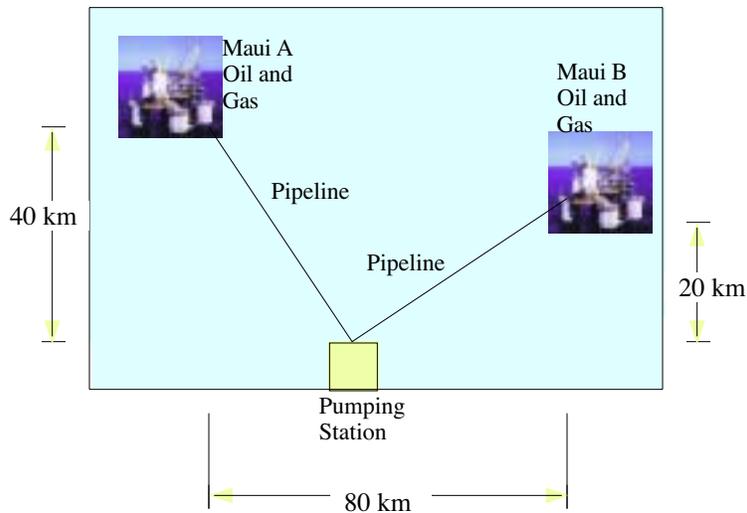
The lab manual says that (as a precaution) a metal rod should be placed in the cylinder so that it cannot be over-compressed (and shatter).



Note that the volume of a cylinder is $\pi r^2 l$, where r is the radius, and l is the length of the portion of the cylinder that is filled with gas.

- (a) What kind of function (linear exponential or power) does the best job of representing the relationship between the volume of the gas-filled portion of the cylinder (independent variable) and the pressure in the gas-filled portion of the cylinder (dependent variable)?
- (b) Find an equation for the pressure as a function of the length of the gas-filled portion of the cylinder.
- (c) The walls of the cylinder can withstand a pressure of up to 30 atmospheres. How far from the non-plunger end of the tube should the rod be placed?
- (d) During part of the experiment, the students have to push the plunger into the cylinder at a steady rate of 0.5 centimeters per second. How rapidly is the pressure changing at the instant of time when the length of the gas-filled cylinder is 15 cm?

8. The diagram given below shows part of the province of Taranaki on the west coast of the North Island of New Zealand. Taranaki has been nicknamed the “Energy Province” of New Zealand because reserves of petroleum and natural gas have been located there. The two main natural gas fields are called “Maui A” and “Maui B.” Spurred by the “oil shocks” of the late 1970’s the New Zealand government began to set up the pipelines and stations needed to extract natural gas and oil from the Maui fields. The most popular plan was to build two pipelines that ran to a central pumping station on shore.



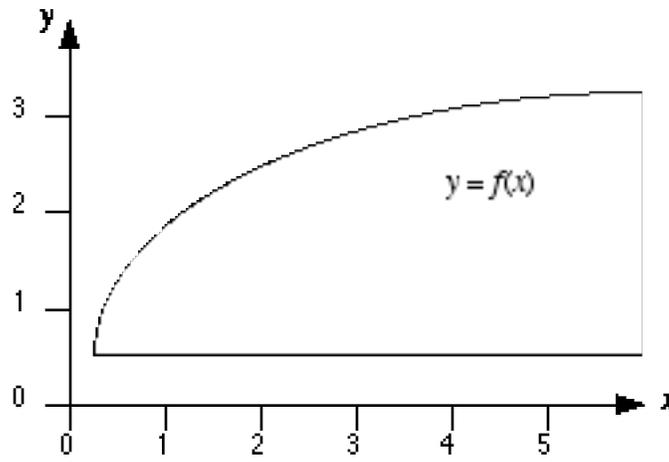
The Maui A platform was located 40 km off the coast and the Maui B platform was 20 km off the coast. The distance between the two platforms (along the coastline) was 80 km. Where was the most cost-effective place to build the pumping station?

9. Sleep deprivation is one of the most well studied impediments to clear reasoning, sound judgment and academic performance. The subject has been widely studied by several branches of the military as soldiers (especially officers) often experience sleep-deprivation during stressful episodes such as combat operations. The military has been particularly interested in studying how sleep deprivation affects soldiers' ability to perform fairly complex information-processing tasks not unlike the problems that college students encounter on final examinations. The results from a very informal study conducted with a group of college students are shown in the table below.

Hours of Sleep	8	4	7	2	8	5
Score on Exam	96	24	74	6	92	40

- Find the equation of the linear function that best represents this data.
- Find the equation of the exponential function that best represents this data.
- Find the equation of the power function that best represents this data.
- Which of the three functions best represents the data ? Explain your answer.
- A student needs to get an 84 on the exam to pass the course. How many hours sleep should that student try to get?

10. Consider the function $y = f(x)$ graphed below. For each of the following pairs of numbers, decide which is larger.



- (a) $f(3)$ or $f(4)$?
- (b) $f(3) - f(2)$ or $f(2) - f(1)$?
- (c) $\frac{f(2) - f(1)}{2 - 1}$ or $\frac{f(3) - f(1)}{3 - 1}$?
- (d) $f'(1)$ or $f'(4)$?