Math Xa **Fall 2002**

Solutions: Final Exam – Set #1

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The domain of $g(x)$ is the interval [-4, 4].
1.(b) The x-coordinates of the points where the derivative of $f(x)$ is equal to zero are: x = -3, -2, -1, 0, 1, 2, 3. The derivative of $f(x)$ is not defined at either of the points x = -4 or x = 4 and so cannot be equal to zero at these points.
1.(c) $g'(x) = 2\cdot f(x)\cdot f'(x) - 2\cdot f'(x) = 2\cdot f'(x)\cdot\left[f(x)-1\right]$.
1.(d) Based on the answer to part (c), $g'(x) = 0$ when either $f'(x) = 0$ or when $f(x) = 1$. The points at which $g'(x) = 0$ are: x = -3.5, -3, -2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5, 3 and 3.5.
1.(e) Classifying the critical points from Part (d) is not all that easy, but you can do it with just the information contained in the graph of $y = f(x)$ and the equation for $g'(x)$. The method for classifying each critical points (as a maximum, a minimum or neither) that is used here is to look at the sign of $g'(x)$ just to the left and just to the right of the critical point. For example, for the critical point located at x = -3.5:

> **Just to the left of x = -3.5:** $f(x)$ is slightly larger than 1, so $[f(x) - 1]$ is positive. However, the graph of $y = f(x)$ is decreasing so $f'(x) < 0$. Therefore $g'(x)$ is equal to a positive times a negative and $g'(x)$ is negative just to the left of x = -3.5.
>
> **Just to the right of x = -3.5:** $f(x)$ is slightly smaller than 1, so $[f(x) - 1]$ is negative. The graph of $y = f(x)$ is still decreasing so $f'(x) < 0$. Therefore $g'(x)$ is equal to a negative times a negative, and so $g'(x)$ is positive just to the right of -3.5.

Therefore, $g(x)$ has a local minimum at x = -3.5. The classification of each critical point is given in the table below.

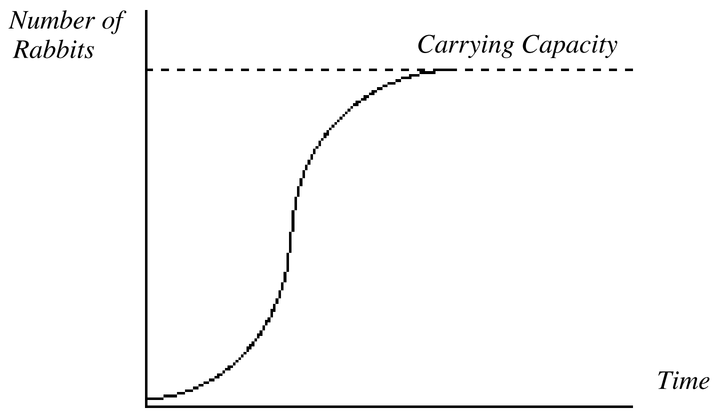
Point	Classification	Point	Classification
x = –3.5	Local minimum	x = 0.5	Local minimum
x = –3	Local maximum	x = 1	Local maximum
x = –2.5	Local minimum	x = 1.5	Local minimum
x = –2	Local maximum	x = 2	Local maximum
x = –1.5	Local minimum	x = 2.5	Local minimum
x = –1	Local maximum	x = 3	Local maximum
x = –0.5	Local minimum	x = 3.5	Local minimum
x = 0	Local maximum		

2.(a) No the number of rabbits cannot be a linear function as the slope is not constant. Between 1859 and 1866 the slope is 2032.71 rabbits per year whereas between 1866 and 1869 the slope is 672,915.67 rabbits per year.
2.(b) No, the number of rabbits is not an exponential function as you get different values for the growth factor depending on what data points you use to calculate it. For example, if you use the points (0, 24) and (7, 14,253) you get a growth factor of B = 2.490245414. If you use the points (7, 14,253) and (10, 2,033,000) then you get a growth factor of B = 5.224888551.
2.(c) No, the number of rabbits is not a power function as the graph of rabbits versus time does not pass through the origin, (0, 0).
2.(d) Of the three possibilities, exponential regression appears to give the correlation coefficient that is closest to 1. If N = number of rabbits and T = number of years since rabbits introduced then the exponential equation is:

$$N = 17.86013243033 \cdot (2.990772231941)^T.$$

2.(e) Substituting $T = 143$ into this equation gives approximately $1.88 \cdot 10^{69}$ rabbits.

2.(f) Evaluating the exponential function for $T = 143$ gives about $1.9 \cdot 10^{69}$ rabbits. At a mass of only 1 kg per rabbit, this means that there is 10^{29} times more mass (in the form of rabbits) roaming Mr. Austin's property than there is mass in the sun! The estimate given by the exponential function is therefore much too high. A more realistic scenario for the growth of the rabbit population could be that to begin with their numbers increased in a way that was approximated by exponential growth, but eventually leveled off due to limited space and resources at the "carrying capacity" of Mr. Austin's property. A graph of population versus time that shows this kind of growth is given below.



3.(a) $y = k \cdot (x + 3)(x - 1)(x - 4)$ where k is a negative number.

3.(b) $y = k \cdot (x + 3)(x - 1)(x - 4)$ where k is a positive number.

3.(c) $y = k \cdot (x + 2)(x - 1)(x - 3)(x - 4)$ where k is a positive number.

3.(d) $y = k \cdot (x + 2)(x - 1)(x - 3)^2$ where k is a negative number.

4.(a) The difference quotient is: $\dfrac{g(1+h) - g(1)}{h} = \dfrac{\frac{f(1+h)}{1+h} - \frac{f(1)}{1}}{h}$.

4.(b) Simplifying this difference quotient: $\dfrac{\frac{f(1+h)}{1+h} - \frac{f(1)}{1}}{h} = \dfrac{f(1+h) - (1+h) \cdot f(1)}{h \cdot (1+h)}$

Simplifying further gives:

$$\frac{f(1+h) - (1+h) \cdot f(1)}{h \cdot (1+h)} = \frac{1}{1+h} \cdot \frac{f(1+h) - f(1)}{h} - \frac{1}{1+h} \frac{h \cdot f(1)}{h}$$

4.(c) Taking the limit of this as $h \to 0$ gives: $g'(1) = f'(1) - f(1) = 1 - 1 = 0$.

4.(d) Using the quotient rule: $g'(1) = \dfrac{f'(1) \cdot 1 - 1 \cdot f(1)}{1^1} = 0$.

5.(a) Defining $f(x)$ is pieces: $f(x) = \begin{cases} -1 & , x < 1 \\ x - 2 & , x \geq 1 \end{cases}$.

5.(b) The derivative of $g(x)$ is given by the Product Rule: $g'(x) = f'(x) \cdot e^x + f(x) \cdot e^x$. So, in order for the derivative of $g(x)$ to be defined, both $f(x)$ and the derivative $f'(x)$ must both be defined. $f(x)$ is always defined, but the derivative $f'(x)$ is not defined at $x = 1$, as there is a "sharp corner" there. Therefore, the domain of the derivative $g'(x)$ consists of all real values of x except $x = 1$.

5.(c) Defining the derivative $g'(x)$ in pieces: $$g'(x) = \begin{cases} -e^x & ,x < 1 \\ e^x + (x-2)\cdot e^x & ,x > 1 \end{cases}$$

5.(d) Defining the derivative $g''(x)$ in pieces: $$g''(x) = \begin{cases} -e^x & ,x < 1 \\ 2e^x + (x-2)\cdot e^x & ,x > 1 \end{cases}$$

5.(e) The original function $g(x)$ will be concave up when the second derivative is positive and concave down when the second derivative is negative. Based on the equation for the second derivative found in Part (d), the original function $g(x)$ will be concave down when $x < 1$ and concave up when $x > 1$.

6.(a) No the curve cannot be the graph of a function. This is because the curve fails the vertical line test.

6.(b) $\dfrac{dy}{dx} = \dfrac{-(x-1)}{y}$.

6.(c) $\dfrac{dy}{dx} = \dfrac{-(x-1)}{\sqrt{1-(x-1)^2}}$.

6.(d) The two equations aren't really the same. The equation obtained in Part (b) works for any point (x, y) that lies on the curve (except (0, 0) and (2,0)) regardless of whether the point lies above the x-axis or below the x-axis. The equation for the derivative obtained in Part (c) only works when the point lies above the y-axis.

7.(a) Based on correlation coefficients, a power function does the best job.

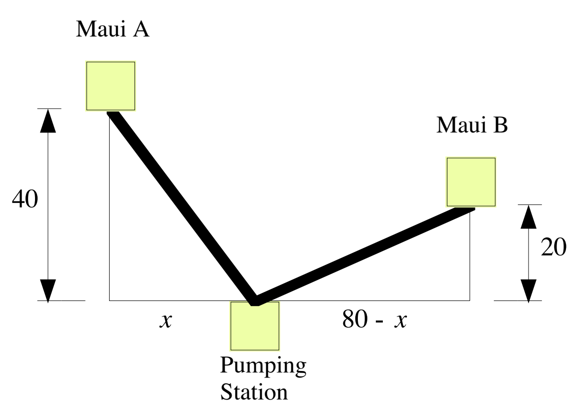
7.(b) Let V = volume of gas (in cubic cm) and P = pressure of gas (in atmospheres). Then from power regression on a calculator:

$$P = 636.56 \cdot V^{-1.42076}.$$

7.(c) The rod should be positioned about 2.7332 cm from the non-plunger end of the cylinder.

7.(d) At that instant of time, the rate of change of the pressure was about +0.12645 atmospheres per second.

8. The most cost-effective solution will probably be the one that involves the shortest possible pipelines. Therefore, the quantity that you should try to minimize here is the total length of the pipeline. If the variable x represents the distance (along the coastline) between the Maui A platform and the pumping station, then the situation can be represented as:



Using the Pythagorean Theorem the total length of the pipeline required to connect both platforms to the pumping station can be expressed as a function of x:

$$L(x) = \sqrt{40^2 + x^2} + \sqrt{20^2 + (80 - x)^2}\,.$$

Differentiating with respect to *x*:

$$L'(x) = \frac{x}{\sqrt{40^2 + x^2}} - \frac{(80 - x)}{\sqrt{20^2 + (80 - x)^2}}\,.$$

Setting the derivative equal to zero and solving for x gives: $x = 160/3$ km. Therefore, in order minimize the total length of the two pipelines, the pumping station should be built 160/3 km from the Maui A platform (distance measured along the Taranaki coast).

9.(a) $s(t) = 15.295*t - 31.341$ (correlation coefficient = 0.99032288)
9.(b) $s(t) = 3.449*e\text{^}(0.43*t)$ (correlation coefficient = 0.97573011)
9.(c) $s(t) = 1.539*t\text{^}1.9872$ (correlation coefficient = 0.99950402)
9.(d) The power function - it has the correlation coefficient that is closest to 1.
9.(e) About 7.48 hours.

10.(a) $f(4)$.
10.(b) $f(2) - f(1)$
10.(c) $(f(2) - f(1))/(2 - 1)$
10.(d) $f'(1)$