Math Xa **Fall 2001**

Practice Problems: Final Exam – Set #2

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.

2. The test will include twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test.

5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

7. Good sources of help:

   - Section leaders’ office hours (posted on Xa web site).
   - Math Question Center (during the reading period).
   - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.

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1. Figures 1 and 2 show the graphs of the derivative and the second derivative of a function, $f(x)$.

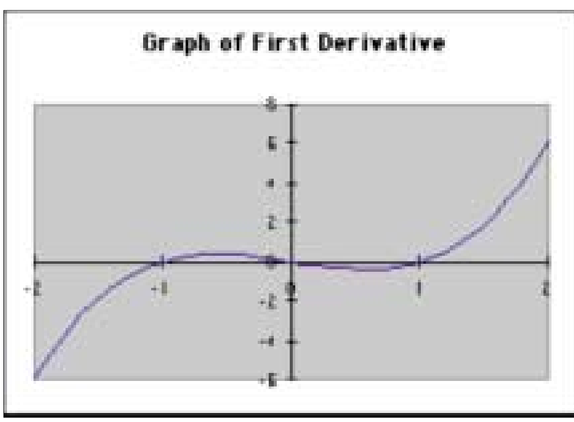

Figure 1: Graph of first derivative.

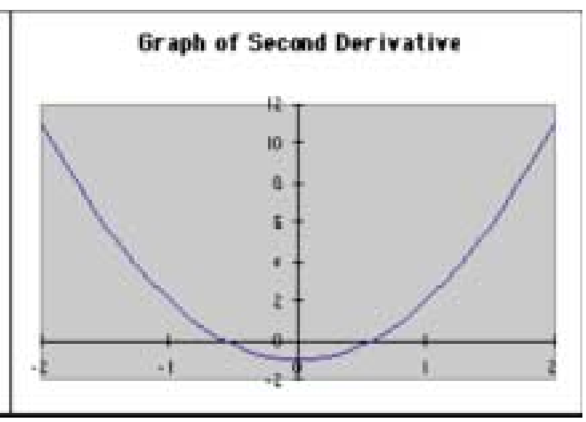

Figure 2: Graph of second derivative.

(a) Locate the x-coordinates of any critical points of the function f. Use the information provided by Figure 2 to classify the critical points of f as local maximums, local minimums or neither.

(b) Locate the x-coordinates of any inflection points of the function f. Explain how you can tell that the points you have identified truly are inflection points, rather than just points where the second derivative is zero but the concavity of the original function does not change.

(c) Sketch graphs showing the original function, first derivative and second derivative of a function that has a point at which the second derivative is equal to zero, but the original function does not have a point of inflection there. (Note that this problem is making the observation that places where the second derivative equals zero usually correspond to places where the original function has an inflection point – but not always.)

2. In this problem, $f(x)$ and $g(x)$ are functions that have derivatives. All that you can assume about them is

- $f'(2) = 7$
- $f(2) = 2$
- $g'(2) = -4$
- $g(2) = 18$.

Use the information given about $f(x)$ and $g(x)$ to calculate the derivatives of the functions given below.

(a) $p'(2)$, where $p(x) = \dfrac{f(x)}{g(x)}$.

(b) $q'(2)$, where $j(x) = f(x) \cdot [g(x)]^2$.

(c) $j'(2)$, where $j(x) = [f(x) + g(x)]^4$.

(d) $m'(2)$, where $m(x) = g(x) \cdot \ln(f(x))$.

3. Sketch a graph of a single function, $f(x)$, that has all of the following features:

- When $x < 2$, $f'(x) > 0$.
- When $2 < x < 4$, $f'(x) < 0$.
- When $x > 4$, $f'(x) > 0$.
- When $x < 3$, $f''(x) < 0$.
- When $3 < x < 5$, $f''(x) > 0$.
- When $x > 5$, $f''(x) < 0$.
- $f(0) = 4$.

NOTE: You do *not* need to come up with an equation for $f(x)$, all you need to do is produce a graph of $y = f(x)$ that shows all of the features listed above.

4. Each of the following expressions relates x and y using an equation that is quite difficult to rearrange into a format where y appears as the subject of the equation. In each case, find an equation for the derivative $\dfrac{dy}{dx}$. Your equation may involve both x and y, but the derivative $\dfrac{dy}{dx}$ should be the subject of the equation (and not appear anywhere else in the equation).

(a) Find an expression for $\dfrac{dy}{dx}$ when: $x + x \cdot y + y^2 = 1$.

(b) Find an expression for $\frac{dy}{dx}$ when: $x \cdot y + \ln(y) = 3$.

(c) Find an expression for $\frac{dy}{dx}$ when: $e^y + e^x = 10$.

5. The equation,

$$x^2 + 2x \cdot y + 3y^2 = 2$$

defines the elliptical curve pictured in Figure 3 below.

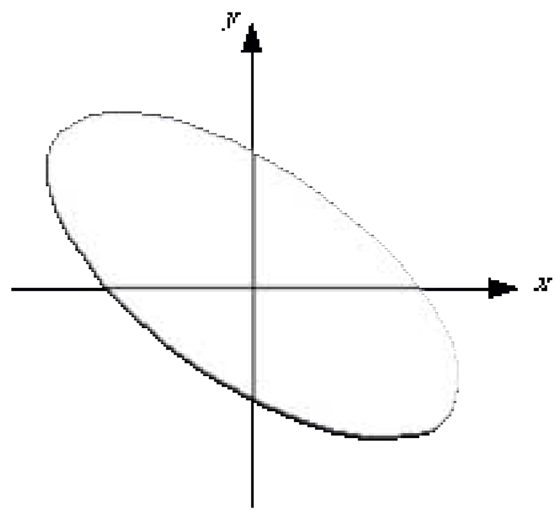

Figure 3.

(a) Find an equation for $\frac{dy}{dx}$.

(b) Verify that the point $(x, y) = \left(0, \sqrt{\frac{2}{3}}\right)$ lies on the curve, and calculate the equation of the tangent line to this point on the curve.

(c) Mark the points on Figure 3 at which $\frac{dy}{dx} = 0$.

The point of the last three parts of this problem is to calculate the exact values of the *x*-coordinates of the points that you have just marked on Figure 3 – that is, the points at which the derivative is equal to zero.

(d) Start with the equation for the derivative that you calculated in Part (a). Set this derivative equal to zero and re-arrange to create a formula for *y* as a function of *x*. At any point on the elliptical curve that satisfies this equation, the derivative will be equal to zero.

(e) Use the equation that you found in Part (d) to eliminate *y* from the equation that defines the elliptical curve:

$$x^2 + 2x \cdot y + 3y^2 = 2.$$

When you have completed this step of the problem, you will be left with n equation that includes only x's, powers of x and constant numbers.

(f) Solve the equation that you obtained in Part (e) to find the precise values of the x-coordinates of the points at which the derivative is equal to zero.

6. The ponderosa pine (*Pinus ponderosa*) is a species of tree that grows in the western region of North America at altitudes of 1000-4000 feet. The average, mature ponderosa pine has a diameter (at its base) of about 3.5 feet and stands about 165 feet tall. The largest ponderosa pine on record stood about 262 feet tall and had a diameter of eight and half feet[1]

The ponderosa pine is much sought-after for harvesting because of its abundant, clear wood, which is particularly desirable for construction. A mature ponderosa pine is normally milled into lumber. The amount of usable wood in a tree is quite well predicted by the diameter of the tree at its base.

One relationship[2] that is used to predict the amount of usable wood from a given tree is given below,

$$g(x) = 0.0039x^{3.137}$$

where: x = base diameter in inches, and,
$g(x)$ = usable wood volume in thousands of cubic inches.

Table 1 shows data relating the age and diameter of a group of ponderosa pines.

Age (years)	4	5	8	8	10	12
Diameter (inches)	2.0	2.0	5.0	7.5	8.8	12.3

Table 1: Age and base diameters[3] for a sample of Ponderosa Pine.

(a) Use the data given in Table 1 to plot a graph showing the diameter of a tree (in inches) versus the age of the tree (in years). What sort of function would do a reasonable job of representing this relationship? Find an equation for diameter as a function of time.

(b) Combine the equation that you found in Part (a) with the equation for the function $g(x)$ to create a function that gives the amount of usable lumber in a tree as a function of the age of the tree.

(c) Find an equation for the derivative of your function from Part (b). Evaluate your derivative for a tree that is four years old and give a practical interpretation (intelligible to a person who has not studied calculus) of the number that you obtain.

(d) The oldest ponderosa pine ever recorded[4] was 600 years old. Use this information and the equations that you have found in Parts (b) and (c) to find the *problem domains* of the function from Part (b) and the derivative from Part (c). In common sense terms would you expect the same equations to apply throughout the entire life of the tree?

(e) According to research[5] on the growth patterns in ponderosa pine, a *growth rate* of a typical ponderosa pine follows the following pattern:

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[1] Source: University of California, Division of Agriculture and Natural Resources. http://www.cnr.berkeley.edu/
[2] Source: COMAP, Inc. (2000) "Mathematics: Modeling Our World. Volume IV." New York: W. H. Freeman and Company.
[3] Source: COMAP, Inc. (2000) "Mathematics: Modeling Our World. Volume IV." New York: W. H. Freeman and Company.
[4] Source: University of California, Division of Agriculture and Natural Resources. http://www.cnr.berkeley.edu/
[5] Sources: P.T. Oester, W. Emmingham, P. Larson and S. Clements. "Performance of ponderosa pine seedlings under four herbicide regimes in northeast Oregon." *New Forests*, **10**(2): 123-131, 1995.

• While a seedling (0-4 years old) the tree grows about 0.050 (thousands of cubic inches) of usable wood per year.
• When the tree matures (4-100 years old) the growth rate of the tree is quite well described by the equation that you found in Question 3.
• When the tree is over 100 years old, it grows about 1.83 (thousands of cubic inches) of usable wood each year.

Using this information, express both the *growth rate* a ponderosa pine as a piece-wise defined functions of the age of the tree. Once you have found an equation for the *growth rate* use it to complete the table given below.

Age of Tree (years)	Amount of usable Lumber (thousands of cubic inches)	Growth rate (in thousands of cubic inches per year)	Amount of growth In next 20 years (thousands of cubic inches)	New amount Of usable Lumber (thousands of cubic inches)
t=0				
t=20				
t=40				
t=60				
t=80				
t=100				
t=120				
t=140				

7. The diagrams shown below give the slope fields of the following four different differential equations (i.e. equations that define the instantaneous rate or derivative of a function):

(a) $$\frac{dy}{dt} = -0.1 \cdot y(t) \cdot \left[y(t) - 3\right]^2$$

(b) $$\frac{dy}{dt} = 0.1 \cdot y(t) \cdot \left[y(t) - 3\right]^2$$

(c) $$\frac{dy}{dt} = \left[y(t) - 2\right] \cdot \left[y(t) + 2\right]$$

(d) $$\frac{dy}{dt} = \left[y(t) - 2\right] \cdot y(t) + 2$$

In each case, decide which slope field (I-IV) corresponds to which differential equation (a-d). When you have matched the differential equations to the slope fields, use the slope fields to sketch a graph of the solution of each differential equation that obeys the initial condition:

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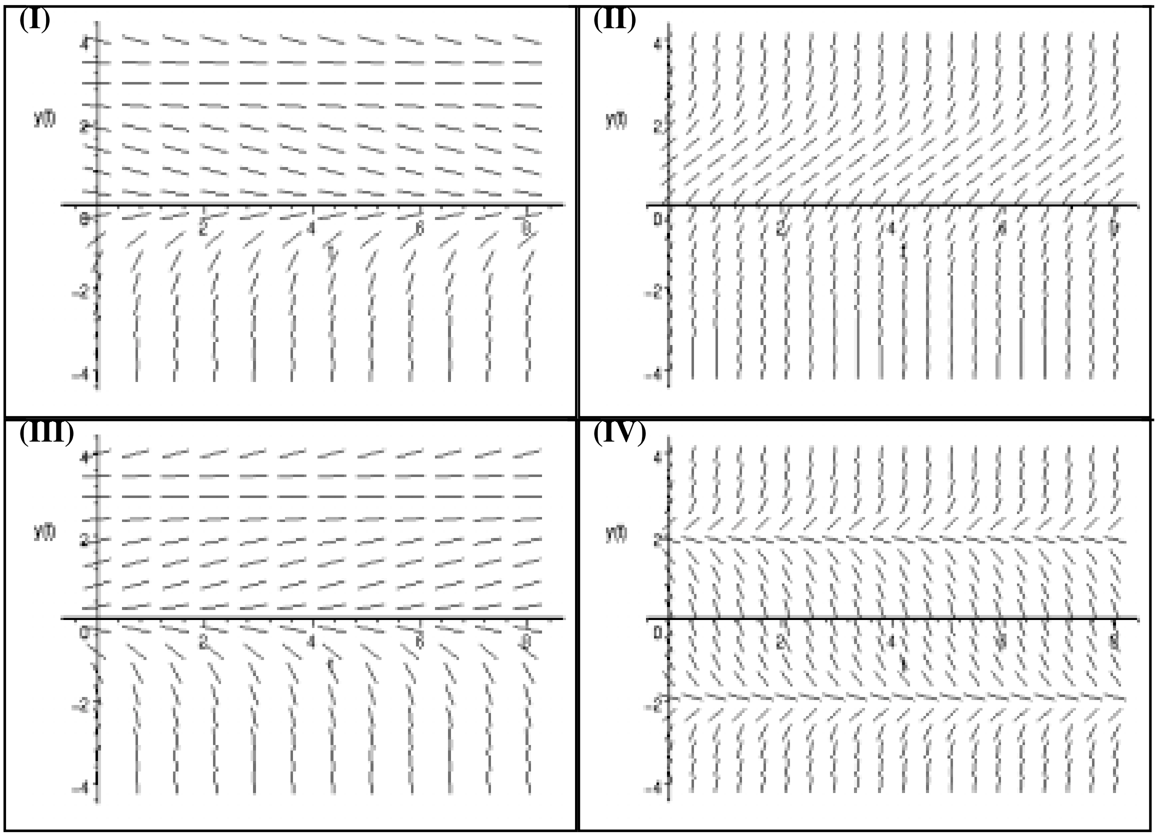
and: M.R. Kaufmann. "To live fast or not: Growth, vigor and longevity of old-growth ponderosa pine and lodgepole pine trees." *Tree Physiology*, **16**(1): 139-144, 1996.

$$y(0) = 1.$$



8. After an unsuccessful campaign to market oil in long, rolled-up tubes (called "Coil 'O' Oil"), a company resorts to selling their oil in more conventional, cylindrical containers. The company plans to sell 500 ml cylindrical cans of oil, but the packaging material for the ends of the can costs 2 cents per square centimeter, whereas the material for the sides costs only one cent per square centimeter. If the company only cares about how much the packaging costs, what should the dimensions of their oil can be?

9. Organic matter contains a radioactive isotope of carbon, known as carbon-14. The half life of carbon-14 is 5730 years. A 100g sample of fresh organic matter will normally contain 0.0001 μg of carbon-14. In this problem, you will examine some of the ways in which the smallest mistakes can drastically alter results when exponential functions and logarithms are involved.

(a) Find an equation to describe the amount of carbon-14 (in micrograms) that remains in a 100g sample of organic matter after T years.

(b) Suppose that a 100g sample of organic matter contains 0.0000327 μg of carbon-14. How old is the organic matter?

(c) Now, suppose that you misread the number from Part (b) and used the figure of 0.0000372 μg of carbon-14 instead. How far off would the age of the organic matter be?

(d) Lastly, suppose that your little buddy Barry calculated the age of a sample of organic matter and determined that it was 2000 years old. Looking over Barry's work, you notice that he incorrectly remembered the half life of carbon-14 to be 5370 years and used this figure in his calculations. What is the correct age of the organic matter?

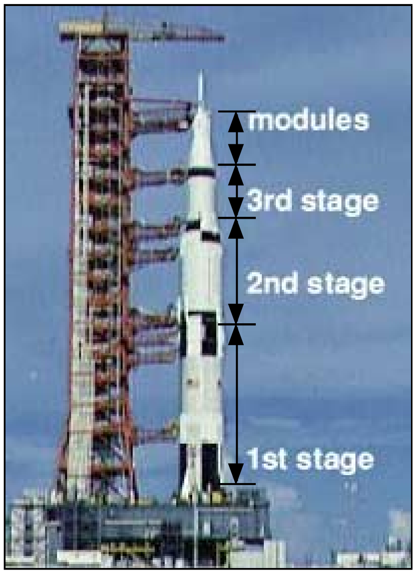


Figure 33.1: The Saturn V launch vehicle. (The three propulsion stages are indicated.)

10. The largest rocket successfully launched was the Saturn V rocket used to carry men to the moon during the late 1960's and early 1970's (see Figure 33.1). The Saturn V consisted of three propulsion stages and a final stage carrying the astronauts and their equipment.

Some of the performance data for the Saturn V rocket are given in the table below.

Stage	Initial mass* (kg)	Fuel consumption (kg per second)	Exhaust velocity (meters per second)	Total burn time (seconds)
First stage	2286217	13360.24	2496.96	161
Second stage	490778	1158.28	4800.22	390
Third stage	119900	253.2	4800.22	421

* The initial mass includes the masses of the modules and all stages that have not yet fired.

(a) Find a collection of functions that give the total mass of the Saturn V rocket at all times between ignition of the first stage and the end of the burn of the third stage.

(b) The velocity of a single-stage rocket at a time *T* seconds after ignition is given by the equation:

$$v(t) = v_0 + v_E \cdot \ln\left(\frac{M_0}{M(t)}\right)$$

where v_0 is the velocity at the time of ignition, v_E is the exhaust velocity and M_0 is the initial mass at the time of ignition. Find a collection of functions that will give the velocity of the Saturn V rocket at all times between ignition and the end of the burn of the third stage.

(c) When does the Saturn V rocket reach maximum velocity?