Math Xa **Fall 2002**

Solutions: Final Exam – Set #2

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The critical points are the points where the first derivative of f is equal to zero. These are the places where the graph of the derivative (Figure 1) has an x-intercept. From Figure 1, these x-intercepts occur at $x = -1$, $x = 0$ and $x = +1$.

The second derivative can be used to decide whether a critical point is a local maximum, local minimum or neither. The important thing is the sign (positive or negative) of the second derivative when you evaluate it at a critical point.

Sign of second derivative at critical point	Type of critical point
Positive (+)	Local minimum
Negative (–)	Local maximum
Zero	Neither[1]

Using the graph in Figure 2, you can decide whether the second derivative is positive or negative at each of the critical points, and then deduce whether the critical point is a local minimum or a local maximum. The results are summarized in the table below.

Location of critical point	Approximate value of second derivative at that point	Classification of critical point
x = –1	1.75	Local minimum
x = 0	–1	Local maximum
x = 1	1.75	Local minimum

1.(b) Points where the second derivative is equal to zero often correspond to inflection points on the graph of the original function. Based on Figure 2, the second derivative appears to be equal to zero at approximately $x = -0.5$ and $x = +0.5$.

There are two ways to determine that both of these points really do correspond to inflection points on the graph of the original function, $y = f(x)$:

First argument - based on Figure 1.

Inflection points on the graph of the original function, $y = f(x)$, correspond to places where the concavity of the original function changes. In terms of the derivative, inflection points correspond to places where the derivative changes from increasing to decreasing (or vice versa). Inspecting Figure 1, you can see that at approximately $x = -0.5$, the graph of the derivative changes from increasing to decreasing, signifying a

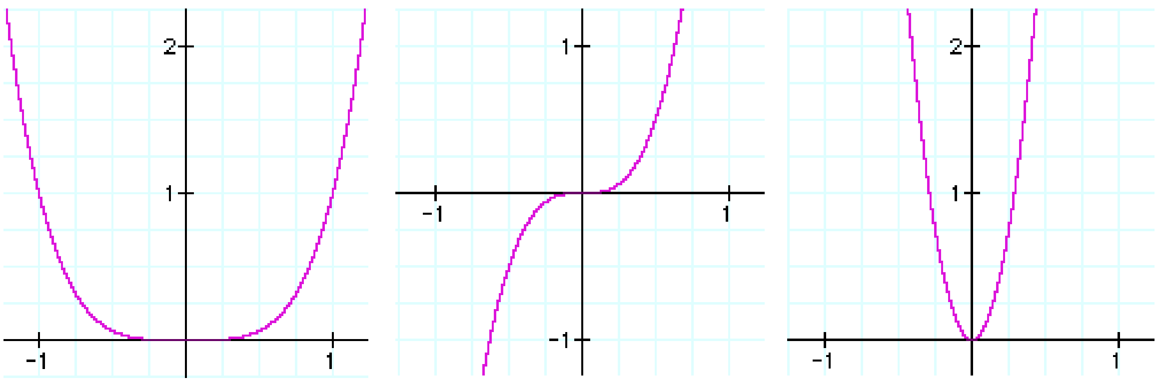
[1] The points where the second derivative is equal to zero often correspond to inflection points on the graph of the original function. It is important to note, however, that just having the second derivative equal zero is not (by itself) a guarantee that the original function will show an inflection point.

point of inflection on the graph of the original function. Likewise, at approximately $x = +0.5$, the graph of the derivative changes from decreasing to increasing signifying a point of inflection on the graph of the original function.

Second argument - based on Figure 2.

Inflection points on the graph of the original function, $y = f(x)$, correspond to places where the concavity of the original function changes. In terms of the second derivative, inflection points correspond to places where the second derivative changes from positive to negative (or vice versa). Inspecting Figure 2 shows that this occurs at approximately $x = -0.5$ and at approximately $x = +0.5$.

1.(c) The key feature that the graph of the second derivative needs to have is a point where the second derivative graph touches the x-axis but doesn't actually cut through the x-axis (like a root of multiplicity two on a polynomial graph). This feature will create a zero on the second derivative graph that does not create a concavity change on the original function graph. Only when the second derivative graph actually cuts through the x-axis does the concavity of the original function graph change, which results in a point of inflection on the original function graph. Perhaps the simplest example of a function that actually does this is the quartic polynomial function $f(x) = x^4$. The original function, first derivative and second derivative graphs shown below all correspond to $f(x) = x^4$.



ORIGINAL FUNCTION **FIRST DERIVATIVE** **SECOND DERIVATIVE**

2.(a) The function $p(x)$ is a quotient of two other functions and is differentiated using the quotient rule.

$$p'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}.$$

Setting $x = 2$ and substituting in the given values for the functions and derivatives:

$$p'(2) = \frac{f'(2) \cdot g(2) - f(2) \cdot g'(2)}{[g(2)]^2} = \frac{(7) \cdot (18) - (2) \cdot (-4)}{[18]^2} \approx 0.4136.$$

2.(b) To differentiate the function $q(x)$ you will need to use both the product rule and the chain rule (to differentiate the composition $[g(x)]^2$). Doing this gives:

$$q'(x) = f'(x) \cdot [g(x)]^2 + f(x) \cdot 2 \cdot g(x) \cdot g'(x).$$

Setting $x = 2$ and substituting in the given values for the functions and derivatives:

$$q'(2) = f'(2) \cdot [g(2)]^2 + f(2) \cdot 2 \cdot g(2) \cdot g'(2) = 1980.$$

2.(c) The function $j(x)$ is an example of a composite function. The "outside" function is the function $k(x) = x^4$, while the "inside" function is $f(x) + g(x)$. Using the Chain rule for differentiating composite functions gives:

$$j'(x) = 4 \cdot [f(x) + g(x)]^3 \cdot (f'(x) + g'(x)).$$

Setting $x = 2$, and using the given values to evaluate this:

$$j'(2) = 4 \cdot [f(2) + g(2)]^3 \cdot (f'(2) + g'(2)) = 96000.$$

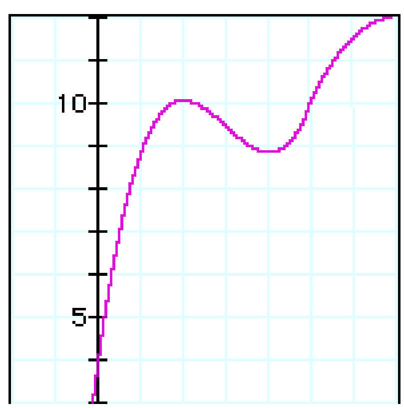
2.(d) Taking the derivative of $m(x)$ will involve the Product rule and the Chain rule. For the Chain rule, the "outside" function is the natural logarithm function and the "inside" function is $f(x)$. Using the Product and Chain rules to differentiate $m(x)$ gives:

$$m'(x) = g'(x) \cdot \ln(f(x)) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x).$$

Substituting $x = 2$ and the given values for f, g and their derivatives:

$$m'(2) = g'(2) \cdot \ln(f(2)) + g(2) \cdot \frac{1}{f(2)} \cdot f'(2) = 60.227.$$

3. There are many possible answers to this problem. What we are looking for here is that you have the right features (y-intercept, increasing/decreasing and concavity) in the right places. The graph of one function that has the required features is shown below.



4.(a) In this problem, x and y are related by the equation:

$$x + x \cdot y + y^2 = 1.$$

Regard x as the variable, and y as a function of x. Differentiating this equation with respect to x gives:

$$1 + y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0.$$

Now, re-arrange to get every term that involves $\frac{dy}{dx}$ on one side of the equation and every term that does not involve $\frac{dy}{dx}$ on the other side of the equation:

$$1 + y = -x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx}.$$

Factor out the $\frac{dy}{dx}$:

$$1 + y = (-x - 2y) \cdot \frac{dy}{dx}.$$

Divide by the factor $(-x - 2y)$ to make $\frac{dy}{dx}$ the subject of the equation:

$$\frac{dy}{dx} = \frac{1 + y}{-x - 2y}.$$

4.(b) In this problem, x and y are related by the equation:

$$x \cdot y + \ln(y) = 1.$$

Regard x as the variable, and y as a function of x. Differentiating this equation with respect to x gives:

$$y + x \cdot \frac{dy}{dx} + \frac{1}{y} \cdot \frac{dy}{dx} = 0.$$

Now, re-arrange to get every term that involves $\frac{dy}{dx}$ on one side of the equation and every term that does not involve $\frac{dy}{dx}$ on the other side of the equation:

$$y = -x \cdot \frac{dy}{dx} - \frac{1}{y} \cdot \frac{dy}{dx}.$$

Factor out the $\frac{dy}{dx}$:

$$y = \left(-x - \frac{1}{y}\right) \cdot \frac{dy}{dx}.$$

Divide by the factor $\left(-x - \frac{1}{y}\right)$ to make $\frac{dy}{dx}$ the subject of the equation:

$$\frac{dy}{dx} = \frac{y}{\left(-x - \frac{1}{y}\right)}.$$

4.(c) In this problem, *x* and *y* are related by the equation:

$$e^x + e^y = 10.$$

Regard *x* as the variable, and *y* as a function of *x*. Differentiating this equation with respect to *x* gives:

$$e^x + e^y \cdot \frac{dy}{dx} = 0.$$

Now, re-arrange to get every term that involves $\frac{dy}{dx}$ on one side of the equation and every term that does not involve $\frac{dy}{dx}$ on the other side of the equation:

$$e^y \cdot \frac{dy}{dx} = -e^x.$$

Divide by the factor e^y to make $\frac{dy}{dx}$ the subject of the equation:

$$\frac{dy}{dx} = \frac{-e^x}{e^y}.$$

5.(a) Differentiating the equation,

$$x^2 + 2x \cdot y + 3y^2 = 2$$

with respect to *x* gives:

$$2x + 2y + 2x \cdot \frac{dy}{dx} + 6y \cdot \frac{dy}{dx} = 0.$$

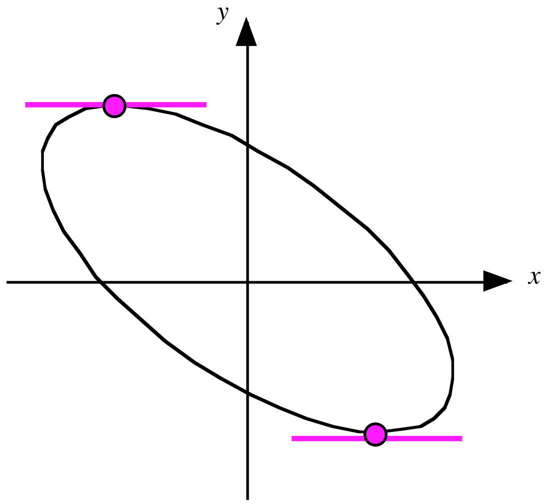
Re-arranging this equation to make $\frac{dy}{dx}$ the subject of the equation gives:

$$\frac{dy}{dx} = \frac{-x - y}{x + 3y}.$$

5.(b) To verify that the point $(x, y) = (0, \sqrt{\tfrac{2}{3}})$ lies on the curve, you simply substitute these values into the equation for the ellipse and make sure that everything comes out to be equal to 2. To find the equation of the tangent line, first you need the slope. This is obtained by substituting $(x, y) = (0, \sqrt{\tfrac{2}{3}})$ into the equation for the derivative from Part (a). This gives: $m = -1/3$. The equation for the tangent line is then:

$$y = \tfrac{-1}{3}x + \sqrt{\tfrac{2}{3}}.$$

5.(c) There are two points on the elliptical curve that would appear to have horizontal tangent lines. Horizontal tangent lines correspond to points at which the derivative is equal to zero. The actual points are indicated in the sketch shown below.



5.(d) Setting the equation for the derivative equal to zero gives:

$$\frac{dy}{dx} = \frac{-x - y}{x + 3y} = 0.$$

Multiplying both sides of this equation by $(x + 3y)$ gives: $-x - y = 0$. Adding y to both sides of this equation then gives the desired result:

$$y = -x.$$

5.(e) Substituting $-x$ for y in the equation defining the elliptical curve,

$$x^2 + 2x \cdot y + 3y^2 = 2,$$

gives:

$$x^2 + 2x \cdot (-x) + 3 \cdot (-x)^2 = 2.$$

Simplifying this slightly gives: $2 \cdot x^2 = 2.$

5.(f) Solving the equation from 5(e) for x gives that the derivative is zero at the points on the elliptical curve where $x = +1$ and where $x = -1$.

6.(a) The plot of the diameter of the tree versus the age of the tree is show in Figure 1 (below).

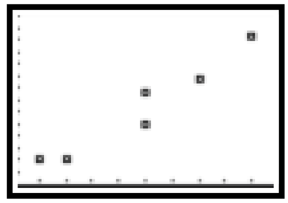


Figure 1.

Figure 1 shows that there is an increasing trend in the data, and that there may be a hint of concavity (concave up) but not much. From the appearance of the plot in Figure 1, a linear, exponential (with growth factor $B > 1$) or power function (with power $p > 1$) might all do reasonable jobs of representing this relationship.

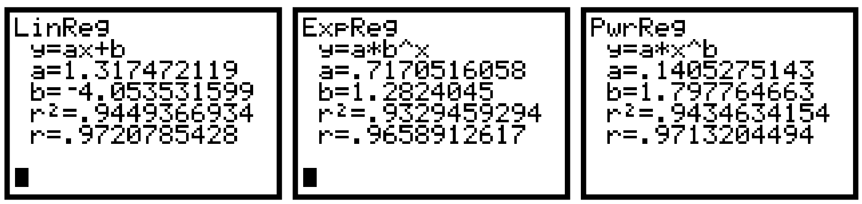


Figure 2.

Trying all of these possibilities (see Figure 2) suggests that a linear function will do the best job of representing the relationship between the age of the tree (T, in years) and the diameter of the tree (D, in inches). The equation for this linear function is:

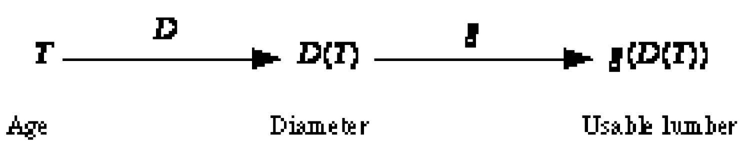
$$D = 1.31747 \cdot T - 4.05353.$$

6.(b) The equation that you are given on the homework assignment:

$$g(x) = 0.0039x^{3.137}$$

where: x = base diameter in inches, and,
$g(x)$ = usable wood volume in thousands of cubic inches.

gives the volume of usable lumber as a function of the diameter of the tree. This is a situation where you can use the output of one function as the input to a second function:



Using the output from the function D as the input to the function g gives:

$$g(D(T)) = 0.0039 \cdot (1.31747 \cdot T - 4.05353)^{3.137}.$$

6.(c) Differentiating the composite function $g(D(T))$ with respect to T using the Chain Rule gives:

$$\left[g(D(T))\right]' = (3.137) \cdot (0.0039) \cdot (1.31747 \cdot T - 4.05353)^{2.137} \cdot (1.31747)$$

Evaluating this when $T = 4$ gives:

$$Derivative \approx 0.0245.$$

A practical interpretation of this number is that between the age of 4 years and the age of five years, the tree increases its amount of useful wood by about 24.5 cubic inches.

6.(d) The *problem domain* is the subset of the *mathematical domain* of the function that represents values of the "input" that "make sense" given the context. First, note that it is impossible to have a tree that has less than zero cubic inches of usable lumber. Therefore, it will only make sense to include values of T in the problem domain that give a positive value of $g(D(T))$. Solving the equation:

$$g(D(T)) = 0.0039 \cdot (1.31747 \cdot T - 4.05353)^{3.137} > 0$$

gives $T > 3.08$.

Second, you are told in the homework that the oldest ponderosa pine ever recorded was 600 years old. This suggests that $T \leq 600$, although it could be argued that the upper limit of the problem domain should be higher as it is theoretically possible at that some time in the history of plant life on earth, there was a living ponderosa pine that was older than 600 years. The problem domain is the set of values of T that lie in the closed interval:

$$[3.08, 600].$$

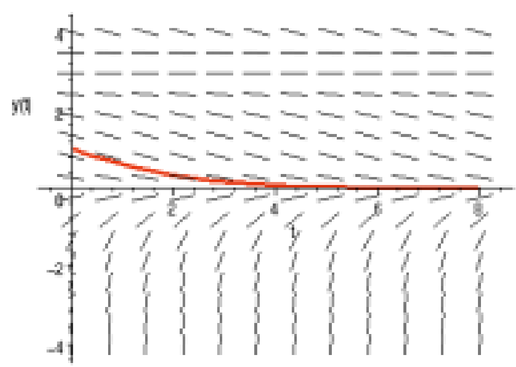
6.(e) Expressing the information about growth rate as a piece-wise function of the age of the tree, T gives:

$$Rate = \begin{cases} 0.050 & ,0 \leq T \leq 4 \\ 0.0161 \cdot (1.31747 \cdot T - 4.0535)^{2.137} & ,4 < T < 100 \\ 1.83 & ,100 \leq T \leq 600 \end{cases}.$$

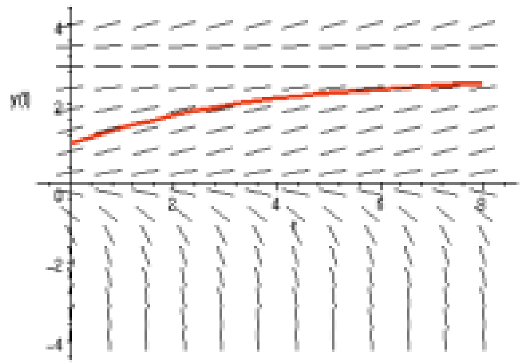
Finally, assuming that a seed that has just germinated does not contain any usable lumber (pretty safe assumption), you can use this piece-wise equation for the rate to determine how the amount of usable lumber in the tree changes with time. (See completed table below.)

Age of tree (years)	Amount of usable lumber (thousands of cubic inches)	Growth rate (thousands of cubic inches per year)	Amount of growth in next 20 years (thousands of cubic inches)	New amount of usable lumber (thousands of cubic inches)
t=0	0	0.050	1	1
t=20	1	12.26	245.19	246.19
t=40	246.19	64.94	1298.84	1545.03
t=60	1545.03	163.78	3275.59	4820.62
t=80	4820.62	311.68	6233.63	11054.25
t=100	11054.25	1.83	36.6	11090.85
t=120	11090.85	1.83	36.6	11127.45
t=140	11127.45	1.83	36.6	11164.05

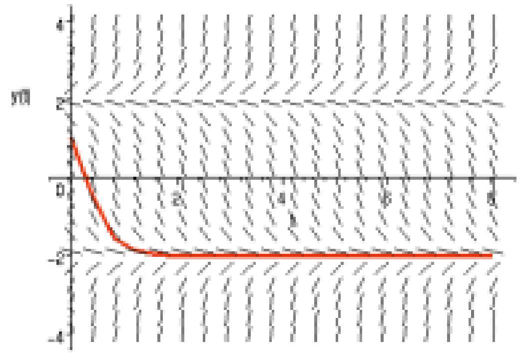
7.(a) $\frac{dy}{dt} = -0.1 \cdot y(t) \cdot [y(t) - 3]^2$. This corresponds to slope field (I). The graph of $y(t)$ that obeys initial value $y(0) = 1$ is shown below.



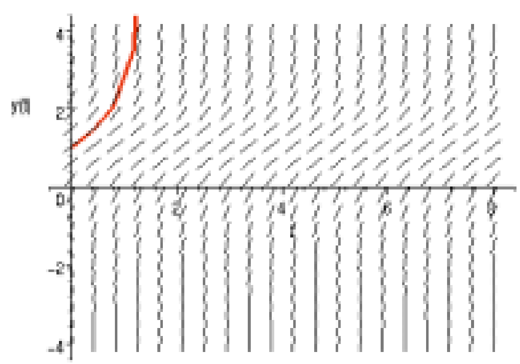
7.(b) $\frac{dy}{dt} = 0.1 \cdot y(t) \cdot \left[y(t) - 3\right]^2$. This corresponds to slope field (III). The graph of $y(t)$ that obeys initial value $y(0) = 1$ is shown below.

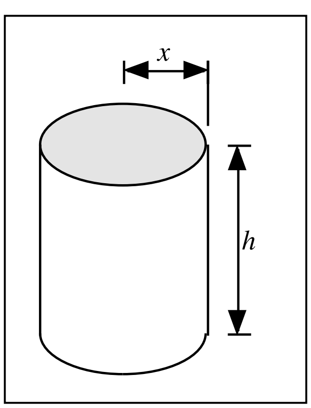


7.(c) $\frac{dy}{dt} = \left[y(t)-2\right]\cdot\left[y(t)+2\right].$ This corresponds to slope field (IV). The graph of $y(t)$ that obeys initial value $y(0) = 1$ is shown below.



7.(d) $\frac{dy}{dt} = \left[y(t)-2\right]\cdot y(t)+2.$ This corresponds to slope field (II). The graph of $y(t)$ that obeys initial value $y(0) = 1$ is shown below.





8. The short answer is: Radius = 3.4139 cm. Height = 13.6556 cm.

Probably the trickiest part of this problem (as is the case with most optimization problems) is setting up the function to maximize or minimize. In this case, the company wants to minimize the cost of its packaging, so we will try to come up with a function that will give the cost of the packaging as a function of either the height (*h*) or the radius (*x*) of the cylindrical package.

Referring to the diagram shown above, the total area of the (two) ends of the oil can is: $2\cdot\pi\cdot x^2$. The total area of the sides of the oil can is: $2\cdot\pi\cdot x\cdot h$. Therefore, the cost (C) of the packaging (in cents) will be:

$$C = 4\cdot\pi\cdot x^2 + 2\cdot\pi\cdot x\cdot h.$$

This has too many variables in it to differentiate, so we will use the fact that the volume of the can has to be 500ml to create an equation to eliminate one of the variables (I will eliminate h but you could eliminate x if you really wanted to and still get the right answer). The volume of the cylinder is: $\pi\cdot x^2\cdot h$. Setting this equal to 500 and rearranging to make h the subject of the equation gives the following.

$$h = \frac{500}{\pi\cdot x^2}.$$

Substituting this into the expression for the cost gives:

$$C = 4\cdot\pi\cdot x^2 + 2\cdot\pi\cdot x\cdot\frac{500}{\pi\cdot x^2} = 4\cdot\pi\cdot x^2 + \frac{1000}{x}.$$

Differentiating gives:

$$C' = 8\cdot\pi\cdot x - \frac{1000}{x^2}.$$

Setting this derivative equal to zero and then solving for x gives that the cost function will have a critical point when:

$$8\cdot\pi\cdot x = \frac{1000}{x^2}$$

$$x^3 = \frac{1000}{8\cdot\pi}$$

$$x = \left(\frac{1000}{8\cdot\pi}\right)^{\frac{1}{3}} \approx 3.414.$$

You can check that this is a minimum by evaluating the derivative slightly to the left and slightly to the right of the point and observe the negative to positive pattern of the derivative, which confirms that the critical point is, indeed, a minimum.

X	3.4	3.414	3.5
Derivative	−1.054	0	6.332

9.(a) $M(T) = (0.0001)\cdot(0.0.9998790392)^T$.

9.(b) The age of the matter will be 9240.41 years.

9.(c) Using the figure of 0.0000372 gives and age of 8174.56. This will be 1065.85 years too young.

9.(d) The correct age will be approximately 2134 years old.

10.(a) The collection of functions is given in the table below. All masses are in kilograms and all times are in seconds.

Time interval (seconds)	Mass (kg)
$0 < T \le 161$	2286217 – 13360.24*T
$161 < T \le 551$	490778 – 1158.24*(T – 161)
$551 < T \le 972$	119900 – 253.2*(T – 551)

10.(b) The collection of functions is given in the table below. All velocities are in meters per second and all times are in seconds.

Time interval (seconds)	Velocity (m/s)
$0 < T \le 161$	$2496.96 \cdot \ln\left(\frac{2286217}{2286217-13360.24*T}\right)$
$161 < T \le 551$	$7060.81 + 4800.22 \cdot \ln\left(\frac{490778}{490778-1158.28*(T-161)}\right)$
$551 < T \le 972$	$19211.03 + 4800.22 \cdot \ln\left(\frac{119900}{119900-253.2*(T-551)}\right)$

10.(c) If you draw a graph of the velocity function, you will see that it is always increasing. Therefore, the maximum value of velocity will be reached at the highest value of T that is allowed. In this case, that is $T = 972$. When $T = 972$ the velocity of the Saturn V rocket is about 29734.9 m/s.