Math Xa **Fall 2001**

Practice Problems: Final Exam – Set #3

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.

2. The test will include twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

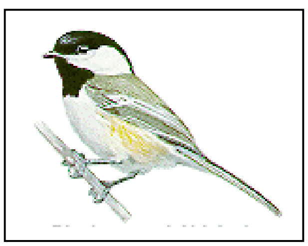
4. You may use your calculator and one page (8" by 11.5") of notes on the test.

5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

7. Good sources of help:

- Section leaders' office hours (posted on Xa web site).
- Math Question Center (during the reading period).
- Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.



1. The Black-capped chickadee[1] (*Poecile atricapillus*) is a small bird that stores seeds (and later finds them again) for food. The table below gives the number of neurons in an average adult Black-capped chickadee hippocampus at different times of the year[2].

Black-capped chickadees are capable of remembering the locations of hundreds of food stores – many more than the average human, but considerably less than the spatial memory "champion" of the animal kingdom, a bird called "Clark's nutcracker" (*Nucifraga Columbiana*). Clark's nutcrackers are able remember the locations of up to 15,000 different food storage sites.

Month	4	6	8	10	12
Number of neurons (hundreds of thousands)	9.2	7.5	9.4	11.4	8.7

(a) Use the axes given below to plot the number of neurons versus the month.

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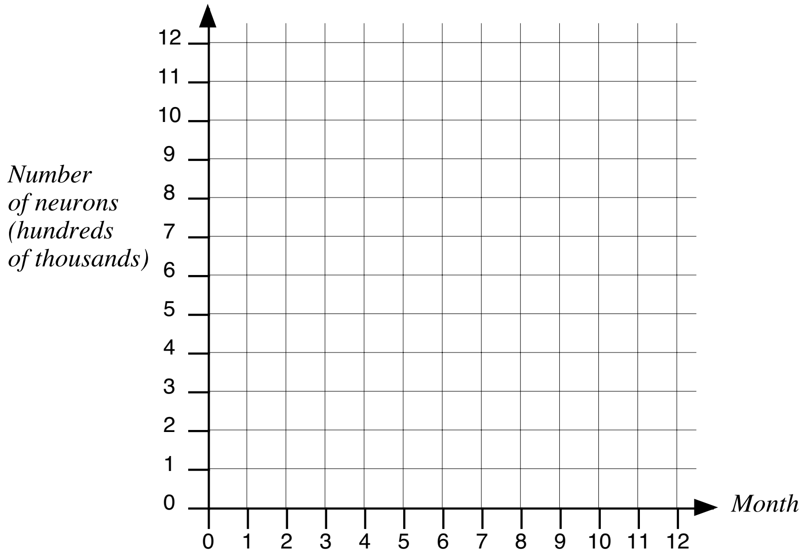
[1] Picture by Larry McQueen. Image source: Cornell Laboratory of Ornithology, http://birds.cornell.edu/

[2] Source: Smulders, T.V., Shiflett, M.W., Sperling, A.J. and DeVoogd, T.J. (2000) "Seasonal changes in neuron numbers in the hippocampal formation of a food-hoarding bird: The Black-capped chickadee." *Journal of Neurobiology*, **44**(4): 414-422.



(b) Which of the functions given below would do a good job of representing the relationship between number of neurons and month?

(c) Approximately how many neurons will an average adult Black-capped chickadee have in it's hippocampus in January (i.e. month 1).

(d) Do you think that the function that you have selected will *always* do a good job of representing the number of neurons in a chickadee hippocampus? Give at least two different reasons for your answer.

2. In this problem the function f will always be the function defined by the equation given below.

$$f(x) = \frac{1}{1+e^{-x}}.$$

(a) Determine the intervals on which f is an increasing function and the intervals on which f is a decreasing function.

(b) Is the inverse of f a function in its own right? If so, find a formula for the inverse of f.

3. "Yankee Home Furnishings" is a small, family-owned furniture company that specializes in handcrafted furniture in a traditional "Shaker" style. Most of their business is with individual customers who want particular items of custom furniture made to their individual specifications. Occasionally, the company manufactures larger orders for corporate customers. In a major deal last year, the company agreed to deliver at least 300 and a maximum of 400 chairs for a major hotel chain. (The hotel chain was not sure of the exact number of chairs that they needed, and the deal was written so that they would get at least 300 chairs and more if they needed them.) The base price was \$90 per chair up to 300 chairs. If the hotel chain wanted more than 300 chairs, then the price would be reduced by 25 cents per chair (on the whole order) for every additional chair over 300 that the hotel chain ordered. What is the largest and the smallest revenue that Yankee Home Furnishings could make from this deal?

4. In 1975 the population of Mexico was about 84 million. In this problem, T will always represent the number of years since 1975 and $M(T)$ will always represent the population of Mexico (in units of millions of people. The equation for $M(T)$ is:

$$M(T) = 84 \cdot (1.026)^T.$$

In this problem $U(T)$ will always represent the population of the United States (in units of millions of people). The equation for $U(T)$ is:

$$U(T) = 250 \cdot (1.007)^T.$$

(a) Find an equation for the instantaneous rate of change of the population of Mexico, $M'(T)$.

(b) How quickly was the population of Mexico changing in 1990 (when $T = 15$)? Give a practical interpretation of this number that someone who had not studied calculus could understand.

(c) Is it ever possible that the population of Mexico will equal the population of the United States? If you believe that this could happen, calculate the year when the two populations will be equal. If you believe that this cannot happen explain why.

5. In this problem, the functions f and g are defined by the equations given below.

$$f(x) = x^4 - x - 1 \qquad\qquad g(x) = \tfrac{1}{2}x^2 + 2$$

In this problem you will also be using a function $h(x)$. All that you may assume about $h(x)$ is that:

- $h(2) = 1$
- $h'(2) = -1$.

(a) Define a new function k by the equation: $k(x) = f(x) \cdot g(x)$. Find an equation for the derivative $k'(x)$.

(b) Define a new function m by the equation: $m(x) = \dfrac{g(x)}{h(x)}$. Calculate the numerical value of the derivative $m'(2)$.

(c) Calculate an approximate value for $m(3)$.

6. In this problem, the function g will always refer to the function defined by the following equation:

$$g(x) = \begin{cases} x^2 & , x < 1 \\ 2x - 1 & , x \geq 1 \end{cases}.$$

In most of the examples or problems that you have investigated, the problem has either been about functions defined in pieces or the limit definition of the derivative but never both. In this problem, you will look at what the value of the derivative is at the boundary where two of the "pieces" of a function meet.

(a) Sketch a graph showing $y = g(x)$ between $x = 0$ and $x = 2$.

(b) Based on the appearance of your graph, what do you think the value of $g'(1)$ should be?

The limit definition of the derivative that you are familiar with mentions only the "overall" limit of the function. As you know, in order for the "overall" limit to exist the left hand limit and the right hand limits must exist. In the next two parts of this question, you will calculate the left and right hand limits of the difference quotient for the function g at the point where $x = 1$.

(c) The appropriate difference quotient for the left hand limit is: $\dfrac{g(1)-g(1-h)}{h}$. Use the equations defining g to calculate this difference quotient, simplify and then take the limit as $h \to 0$.

(d) The appropriate difference quotient for the right hand limit is: $\dfrac{g(1+h)-g(1)}{h}$. Use the equations defining g to calculate this difference quotient, simplify and then take the limit as $h \to 0$.

(e) According to the limits that you have calculated in Parts (c) and (d), does the "overall" limit of the difference quotient exist at $x = 1$? If so, what is the value of $g'(1)$?

7. In this problem, the function $f(x)$ is always the function defined in Figure 1 below.

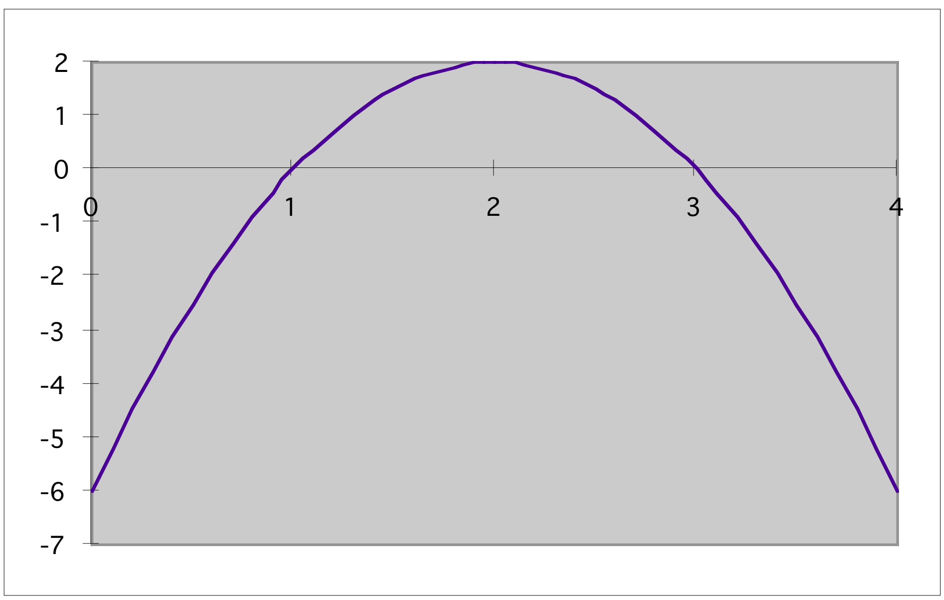


Figure 1: Graph defining $f(x)$.

In this problem, the function $g(x)$ is always the function defined by the equation: $g(x) = 10^x$.

Use the functions $f(x)$ and $g(x)$ to calculate the values of the following derivatives:

(a) $f'(2)$.

(b) $k'(2)$ where $k(x) = \frac{f'(x)}{g'(x)}$.

(c) $m'(1)$ where $m(x) = f(x) \cdot g(x)$.

8. Table 1 (below) shows the percentage of people in the US who lived in rural areas between 1830 and 1960[3].

Year	1830	1840	1850	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950	1960
% rural	91.2	89.2	84.7	80.2	74.3	71.8	64.9	60.4	54.4	48.8	43.9	43.5	36.0	30.1

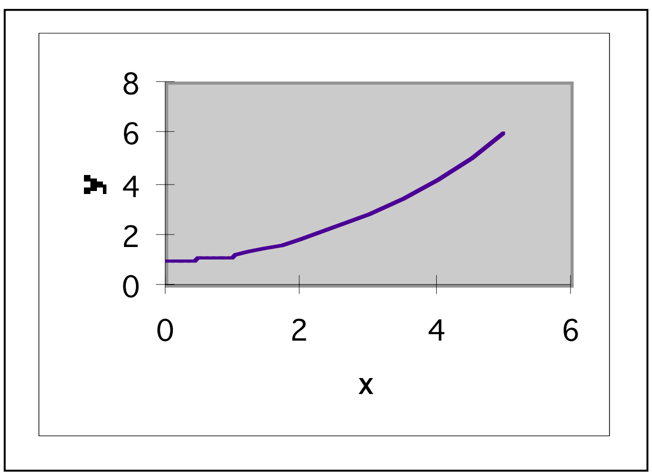
Table 1: Percentage of US population living in rural areas 1830-1960.

(a) Plot a graph showing the percentage of people living in rural areas versus year. What features would a function (increasing/decreasing, concave up/down or no concavity) need to show in order to do a reasonable job of representing the trend(s) in your plot?

(b) WITHOUT USING THE REGRESSION CAPABILITIES OF A CALCULATOR, find an equation that you can use to predict the percentage of people living in rural areas, given the year. Over what period(s) of time do you expect the predictions of your function to closely match the actual values of the data from Table 1?

(c) Use your equation to predict the percentage if people living in rural areas in 2050. Does your answer make sense? Explain why or why not. (Hint: What is a reasonable "problem domain" for the equation you found in Part (b)?)

9. The graph and table shown in Figure 2 correspond to the **same** function. (Use this graph and table to answer all of the questions in this problem.)



x	0	1	5
f(x)	1	1.2	6

Figure 2: Graph and table for Problem 9.

(a) Functions like the one shown above are often exponential functions: $y = A \cdot B^x$.
Use the data in the table to decide whether this function is an exponential function or not.

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[3] Source: US Bureau of the Census, *The Statistical History of the United States* (1976).

(b) Some functions with increasing, concave up graphs are power functions: $y = k \cdot x^p$ when the power p is greater than 1. Use the data in the table to decide whether this function is a power function or not.

(c) One last possibility is that this function could be an exponential function or a power function that has been modified by adding 1 unit to the function, i.e.:

$$y = 1 + k \cdot x^p \qquad \text{OR} \qquad y = 1 + A \cdot B^x$$

Find an equation for this function that exactly matches the data given in the table. As part of your answer, you should show that your equation really does exactly match the values given in the table.

10. Controversial businessman Arnold "Mad Dog" operates a taxi company in the town of Fostoria, Ohio. Mr Johnson's taxi fares are calculated using the rules:

- $1.25 for the first quarter mile (or part thereof).
- $0.50 for each subsequent quarter mile (or part thereof).

(a) Complete the table given below showing the cost of several different taxi rides.

Length of Ride (Miles)	0.66	1.2		7.2	
Cost ($)			13.00		22.95

(b) Write down an equation for cost (in dollars) as a function of the length of the ride (in miles). Your equation should cover all possibilities between 0 miles and 1 mile.