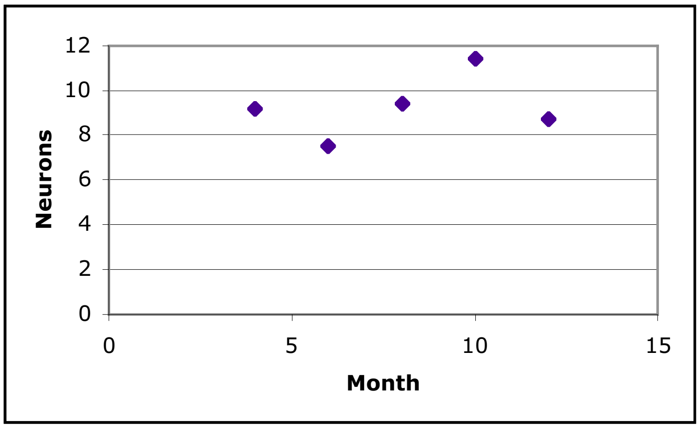
Math Xa **Fall 2002**

Solutions: Final Exam – Set #3

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The graph given below shows the plot of the data.



1.(b) If you imagine a smooth curve going through the points in the above graph, the shape of the smooth curve seems to resemble John Travolta on the "Saturday Night Fever" album cover. Therefore, out of the possibilities listed, the **cubic function** probably does the best job of representing the trend in the data.

1.(c) Plugging x = 1 into the equation for the cubic function that you were given in Part (b) gives 28.258, which represents 2,825,800 neurons.

1.(d) The cubic function probably **does not** do a good job of representing the number of neurons in the chickadee hippocampus at all times. First, when x is a very small number (i.e. less than zero) the cubic function gives extraordinarily large predictions for the number of neurons. This suggests that when they are little chicks, chickadees have massive hippocampuses that contain tens or hundreds of millions of neurons, whereas as adults (when they actually need to use their spatial memories to locate stored food) they have only a few hundred thousand neurons. This could be the case, but I would expect the opposite to be true. Second, when x gets much above 12, the cubic function starts to predict that the chickadee's hippocampus will include a negative number of neurons. This is clearly nonsense, as the least number of neurons that the chickadee could have in its hippocampus should be zero.

2.(a) The derivative of f is: $f'(x) = \dfrac{e^{-x}}{\left(1+e^{-x}\right)^2}$. This quantity is positive, so the original function f is increasing.

2.(b) As f is always increasing, it's graph is always going up when read from left to right. This means that f will pass the horizontal line test. The argument for this goes as follows: suppose that f didn't pass the horizontal line test. Then there would have to be two points of the graph $y = f(x)$ that had the same height. Therefore, the graph of f must have decreased when read from left to right in order to come down to that height a second time. However, the one thing that you definitely know about f is that it is always increasing, so it can't ever have decreased. That means that the assumption that the graph $y = f(x)$ doesn't pass the horizontal line test must be wrong. A formula for the inverse of the function can be written as:

$$x = -1 \cdot \ln(\tfrac{1-y}{y}).$$

3. If the hotel chain takes only 300 chairs then the total revenue will be $27,000. If the hotel chain takes x chairs (where x is between 300 and 400) then the revenue is given by:

$$R(x) = x \cdot (90 - 0.25 \cdot (x - 300)).$$

This problem illustrates the phenomenon that coming up with the function that describes the situation is often the hardest part of an optimization problem. The rationale behind this complicated formula are as follows: First, revenue will be equal to the number of chairs times the price of each chair. Therefore, $R(x) = x \cdot (\text{Price})$. The basic price of each chair is $90. However, this price is lowered by twenty-five cents for every chair (above 300 chairs) purchased. If x chairs are purchased, but the first 300 do not create a discount, then the number of chairs that create discounts will be: $x - 300$. Each of these chairs creates a twenty-five cent discount, so in units of dollars, the total discount on the chair price will be: $0.25 \cdot (x - 300)$. Subtracting this discount from the $90 basic chair price gives the discounted chair price of: $90 - 0.25 \cdot (x - 300)$. Multiplying this price by x, the number of chairs actually sold, gives the revenue, $R(x)$.

Differentiating this function and setting the derivative equal to zero gives $x = 330$. To see this, observe that by FOILing, you can simplify the formula for $R(x)$.

$$R(x) = x \cdot (90 - 0.25 \cdot (x - 300)) = 165 \cdot x - 0.25 \cdot x^2.$$

$$R'(x) = 165 - 0.5 \cdot x$$

Checking the sign of the derivative on each side of $x = 330$ confirms that this is a local maximum.

X	329.9	330	330.1
Derivative	+0.05	0	–0.05

When $x = 330$, the revenue will be $R(330) = \$27,225$. The one other point to check is the end-point $x = 400$. The revenue when $x = 400$ is: $R(400) = \$26,000$. Therefore, the greatest revenue that the furniture company can collect is $27,225 (when 330 chairs are sold) and the lowest is $26,000 (when 400 chairs are sold). Astonishingly, the company collects more revenue when they sell 300 chairs than when they sell 400 chairs. As the cost of producing 400 chairs will almost certainly be significantly higher than the cost of producing 300 chairs, the company will make much smaller profits from selling 400 chairs, as compared with their profits from selling 300 chairs. That is not very good business practice, so they might want to re-think their pricing strategy.

4.(a) Using the short-cut rule for differentiating exponential functions, the instantaneous rate of change of the population of Mexico will be:

$$M'(T) = \ln(1.026) \cdot 84 \cdot (1.026)^T.$$

4.(b) Plugging $T = 15$ into the formula for the derivative calculated in Part (a) gives $M'(15) \approx 3.167$. In practical terms, this means that when time is increased from $T = 15$ to $T = 16$ (i.e. from the year 1990 to the year 1991) the population of Mexico will increase by approximately 3,167,000 people.

4.(c) Yes it is possible that the population of Mexico could equal the population of the United States. To determine when this could happen, we need to solve the following equation for T:

$$84 \cdot (1.026)^T = 250 \cdot (1.007)^T.$$

Rearranging this equation by dividing both sides by 84, and then by $(1.007)^T$ gives:

$$\frac{(1.026)^T}{(1.007)^T} = \frac{250}{84}.$$

This can be simplified a little using the Law of Exponents.

$$\left(\frac{1.026}{1.007}\right)^T = \frac{250}{84}.$$

Taking the natural logarithm of both sides (you could also use the common logarithm here as well – both common and natural logarithm will give the correct answer), and employing the Super Fun Happy Rule gives:

$$T \cdot \ln\left(\frac{1.026}{1.007}\right) = \ln\left(\frac{250}{84}\right).$$

Rearranging this to make T the subject of the equation and evaluating the logarithms on a calculator gives: $T = 58.35$. This corresponds to the year 2033.

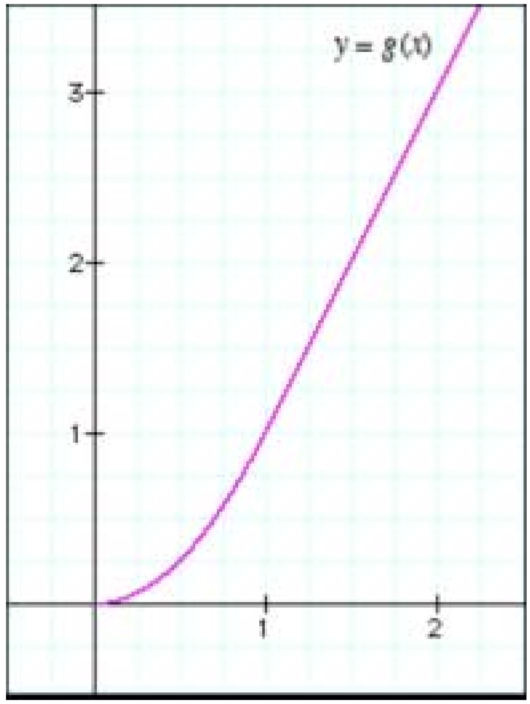
5.(a) As $k(x)$ is a product of two functions, you use the product rule to differentiate it. This gives:

$$k'(x) = \left(4x^3 - 1\right) \cdot \left(\tfrac{1}{2}x^2 + 2\right) + (x) \cdot \left(x^4 - x - 1\right).$$

5.(b) As $m(x)$ is a quotient of two functions, you use the quotient rule to differentiate it. Doing this gives $m'(2) = 6$.

5.(c) You have enough information to work out $m(2)$, as you are told that $h(2) = 1$ and you can use the equation for $g(x)$ to work out that $g(2) = 4$. Therefore, $m(2) = 4$. The derivative $m'(2)$ is approximately how much the function m will change when x is increased from $x = 2$ to $x = 3$. Since $m'(2) = 6$, the function will increase by approximately 6 when x is increased from $x = 2$ to $x = 3$. Therefore, the value of $m(3) \approx 4 + 6 = 10$.

6.(a) The graph of $y = g(x)$ is shown in the diagram below.



6.(b) When $x = 1$, the graph of $y = g(x)$ appears to resemble a straight line. The slope of this line is 2, so I would expect the value of $g'(1)$ to be 2.

6.(c) The left hand difference quotient is:

$$\frac{g(1)-g(1-h)}{h}=\frac{1-(1-h)^2}{h}=\frac{2h-h^2}{h}=2-h.$$

Taking the limit of this as $h \to 0$ gives a limit of 2.

6.(d) The right hand difference quotient is:

$$\frac{g(1+h)-g(1)}{h}=\frac{2+2h-1-1}{h}=\frac{2h}{h}=2.$$

Taking the limit of this as $h \to 0$ gives a limit of 2.

6.(e) As the left hand and right hand limits are both equal to 2, the overall limit exists and is equal to 2.

7.(a) From the graph of $y = f(x)$, $f'(2) = 0$.

7.(b) Using the quotient rule for derivatives,

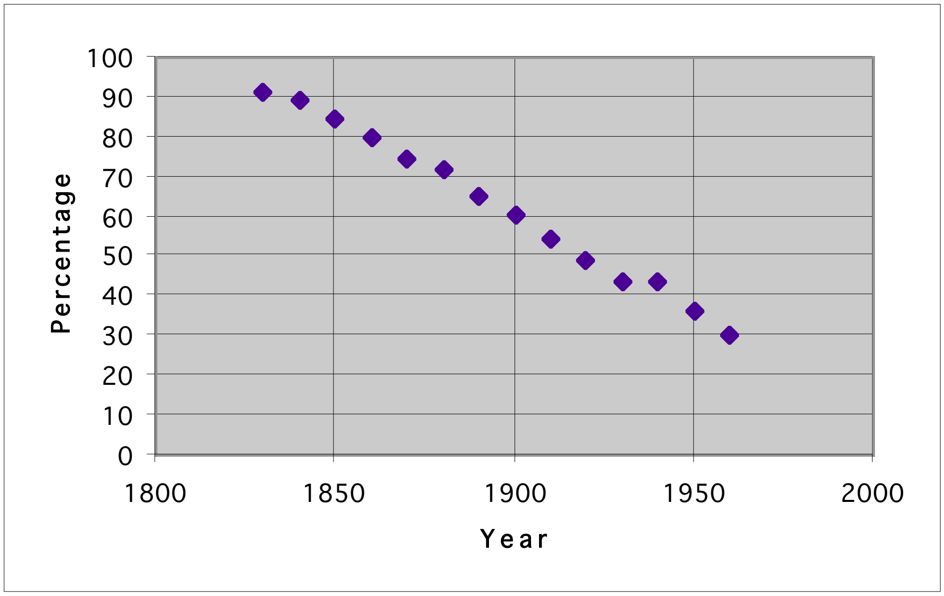
$$k'(2)=\frac{f'(2)\cdot g(2)-f(2)\cdot g'(2)}{[g(2)]^2}=\frac{0*10^2-2*\ln(10)*10^2}{[10^2]^2}=-0.04605.$$

7.(c) In this part of the problem, you have to come up with a value for $f'(1)$. I did this by drawing a tangent line to $x = 1$ on the graph of $y = f(x)$ and then estimating the slope of that tangent line.

Using the product rule for derivatives:

$$m'(1) = f'(1)*g(1) + f(1)*g'(1) = 4*10 + 0*\ln(10)*10 = 40.$$

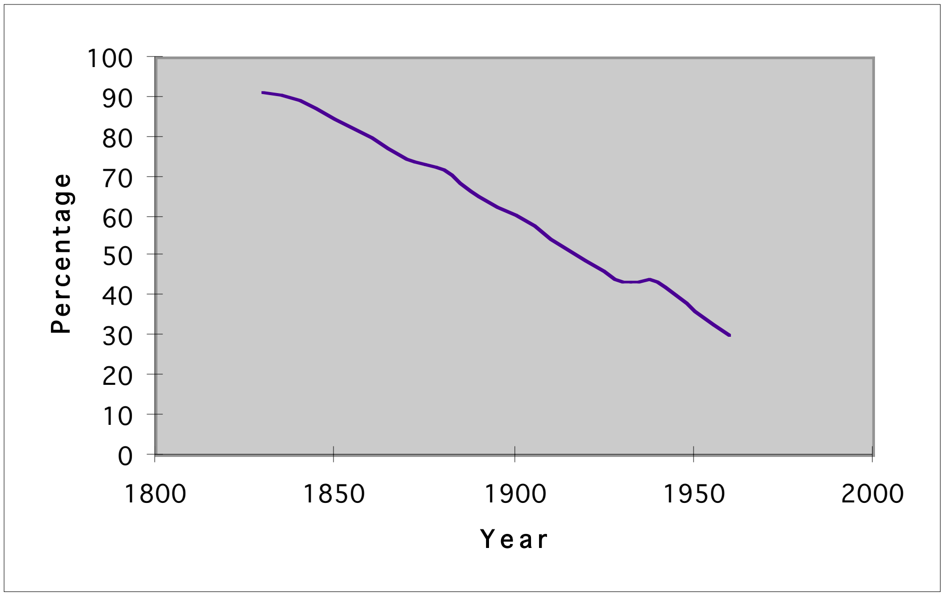
8.(a) The plot of percentage of people living in rural areas versus year is shown below.



A function that was going to represent this relationship would need, first of all, to be **decreasing.** This is because as you read the graph from left to right, you see that the height drops over each decade.

A function that was going to do a reasonable job of representing this relationship would not necessarily have to show a great deal of **concavity**. This is because all of the points appear to lie in a pattern that is quite close to a straight line.

If you were determined to find a function that fit the data points perfectly, then you could sketch a smooth curve through the points – this is shown in below – and then inspect the smooth curve to see where it is concave up and concave down.



Inspection of the smooth curve suggested the intervals given in the table below.

Intervals over which the function should be concave down	Intervals over which the function should be concave up
(1830, 1870)	(1870, 1880)
(1880, 1890)	(1890, 1900)
(1900, 1910)	(1910, 1940)
(1940, 1950)	(1950, 1960)

8.(b) Because the points do not deviate that much from a straight line, a linear function will probably be a reasonable representation of this relationship. Calculating a linear function to do this:

Calculating the slope: $m = \dfrac{30.1 - 91.2}{1960 - 1830} = -0.47.$

Calculating the intercept: $y = m \cdot x + b$

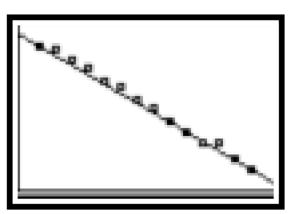
$91.2 = -0.47 \cdot 1830 + b$

$b = 951.30$

Put the equation together: $y = -0.47 \cdot x + 951.30$

where y is the percentage and x is the year.

Plotting both the data and the function on the same set of axes gives the graph shown below.



This shows that the outputs from the function quite closely matches the values of the data points all the way from 1830 to 1960. To see how far outside that range you could possibly go, observe that as the output for the function is the percentage of the US population living in rural areas, it can be safely assumed that the outputs from the function should not rise above 100% and should not fall below 0%. Solving to find the years that the function attains these values:

$$100 = -0.47 \cdot x + 951.30, \quad \text{so that:} \quad x = 1811.28$$

$$0 = -0.47 \cdot x + 951.30, \quad \text{so that:} \quad x = 2024.04$$

The largest set of x-values for which the outputs of the function could be considered reasonable would therefore be the interval: (1811.28, 2024.04).

8.(c) If you substitute $x = 2050$ into the equation from Part (b), then you obtain:

$$y = -0.47 \cdot 2050 + 951.30 = -12.2.$$

This result does not make any sense because you can have a negative percentage of the US population. The reason for this (as indicated by the calculations in Part (b)) is that the "problem domain" of the linear function is, at best, the interval: (1811.28, 2024.04). Since $x = 2050$ lies well outside this problem domain, there's no real reason to think that the output from the function will resemble the actual percentage or people living in rural areas in the US in the year 2050.

9.(a) The most fundamental test for an exponential function is this:

> Calculate the **growth factor** B in the equation for the exponential function:
>
> $$y = A \cdot B^x.$$
>
> If you **always** get the **same value** for B, then the function representing the relationship between x and y is an exponential function.

Using the data given in the homework assignment to calculate B gives the following results:

- **Using the points (0, 1) and (1, 1.2) to calculate *B*:**

$$1.2 = A \cdot B^1$$
$$1 = A \cdot B^0.$$

Dividing these equations and simplifying gives:

$$1.2 = B.$$

- **Using the points (0, 1) and (5, 6) to calculate *B*:**

$$6 = A \cdot B^5$$
$$1 = A \cdot B^0.$$

Dividing these equations and simplifying gives:

$$6 = B^5.$$

Taking the 1/5 power of both sides gives:

$$B = (6)^{1/5} = 1.430969081.$$

The two values of B obtained ($B = 1.2$ and $B = 1.430969081$) are not equal, so the function shown in Figure 1 cannot possibly be an exponential function.

9.(b) A power function that is concave up and increasing must have $k > 0$ and $p > 1$. Under these conditions, if you were to substitute $x = 0$ into the general equation for a power function:

$$y = k \cdot x^p,$$

then you would obtain $y = 0$. However, both the table and the graph from Figure 2 indicated that when $x = 0$, $y = 1$. So, the function cannot possibly be a power function.

9.(c) In the interests of thoroughness, we will examine both cases in turn.

- **Exponential Function with One Unit Added:** $y = 1 + A \cdot B^x$.

From the table in Figure 2, when $x = 0$, $y = 1$. Substituting these values into the equation gives:

$$1 = 1 + A \cdot B^0 = 1 + A,$$

so that $A = 0$. If $A = 0$, then the equation for the exponential function will be:

$$y = 1 + 0 \cdot B^x = 1.$$

This is the equation of a horizontal line with height 1, which does not increase in height and is not concave up. Therefore, the function shown in Figure 2 cannot have an equation of the form: $y = 1 + A \cdot B^x$.

- **Power Function with One Unit Added:** $y = 1 + k \cdot x^p$.

Re-arranging the equation gives that: $y - 1 = k \cdot x^p$. Substituting the point (1, 1.2) into this equation gives:

$$0.2 = k \cdot (1)^p$$

so that $k = 0.2$. Substituting $k = 0.2$ and the point (5, 6) into $y - 1 = k \cdot x^p$ gives:

$$5 = 0.2 * (5)^P.$$

Re-arranging this equation gives: $25 = 5^p$ so that $p = 2$. The equation for the function shown in Figure 2 is therefore:

$$y = 1 + 0.2 * x^2.$$

10.(a) The completed table is shown below.

Length of Ride (Miles)	0.66	1.2	**NOT POSSIBLE**	7.2
Cost ($)	**2.25**	**3.25**	13.00	**15.25**

10.(b) There are many possible, acceptable ways to write down the equation for the function that gives the cost of the taxi ride (and is valid between 0 miles and 1 mile). One option is to write the equation in pieces. This is shown below.

Let C represent the cost of the taxi ride (in dollars) and x represent the length of the ride (in miles).

$$C(x) = \begin{cases} 1.25 & ,0 < x \le 0.25 \\ 1.75 & ,0.25 < x \le 0.5 \\ 2.25 & ,0.5 < x \le 0.75 \\ 2.75 & ,0.75 < x \le 1 \end{cases}$$