Math Xa **Fall 2002**

Practice Problems: Final Exam – Set #4

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.

2. The test will include twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

4. You may use your calculator and one page (8" by 11.5") of notes on the test.

5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

7. Good sources of help:

   - Section leaders' office hours (posted on Xa web site).
   - Math Question Center (during the reading period).
   - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.

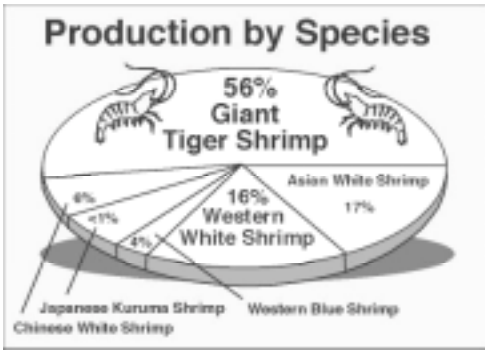

Figure 1: Shrimp production for U.S. market by species. The two most important species are the tiger shrimp and the Asian white shrimp.

1. In the marine context, farming is often called *aquaculture*. At present, the two most valuable farmed organisms are shrimp (various species, the most important probably being tiger shrimp, *Penaeus monodon*, and Asian white shrimp, *Penaeus indicus* – see Figures 1 and 2) and Atlantic salmon (*Salmo salar*). Together, shrimp and salmon account for almost 20% of worldwide aquaculture profits[1]. Most of the farmed shrimp is produced in Asia but consumed in North America and Europe, while the top producers of farmed salmon are Norway, Britain, Canada, the United States and Chile. The global value of the shrimp farming industry is approximately $6 billion annually, while salmon farming generates approximately $2 billion per year.

Shrimp are not particularly efficient creatures when it comes to converting food into shrimp. A typical feed conversion ration (FCR) for tiger shrimp[2] is approximately 4:1. This means that in order to grow one

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[1] Source: Food and Agriculture Organization of the United Nations. 1998. *World Aquaculture Production Statistics.* Rome Italy: United Nations.

[2] Source: Tacon, A. J. G. 1996. *International Aquafeed Directory and Buyer's Guide.* Middlesex UK: Turret Research Associates International.

metric ton of shrimp, four metric tons of food must be given. For Atlantic salmon, the FCR[3] is approximately 2.8:1, making them only about half as efficient as a chicken[4] (FCR = 1.42) but more than twice as efficient as beef cattle[5] (FCR = 6.5).

Figure 2: The two most commercially important shrimp species for the U.S. market. (a) Tiger shrimp (*Penaeus monodon*). (b) Asian white shrimp (*Penaeus indicus*).

Both shrimp and salmon are (in nature) moderately to highly predatory species that are adapted to a diet high in animal proteins. Approximately 33% of the feed typically given to farmed shrimp is composed of fish protein (in the form of fish meal or fish oil), whereas approximately 70% of the feed normally used in salmon farming is composed of fish protein[6]. This fish protein and oil is obtained through the catching and processing of wild fish species that are not yet considered commercially important[7] for direct human consumption (at least in developed nations).

Let t denote the number years since 1982, $S(t)$ denote the worldwide production of farmed shrimp (in units of hundreds of thousands of metric tons).

(a) Worldwide data on shrimp farming[8] can be described using the following equation for the rate of change.

$$\frac{dS}{dt} = -0.083 \cdot S(t) \cdot \left[S(t) - 7\right]$$

Use this equation together with the axes provided to draw a **slope field**.

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[3] Source: Tacon, A. J. G. 1996. *International Aquafeed Directory and Buyer's Guide*. Middlesex UK: Turret Research Associates International.

[4] Source: Agriculture and Agri-food Canada. http://www.agr.gc.ca/cal/epub/1860e/1860-0003_e.html

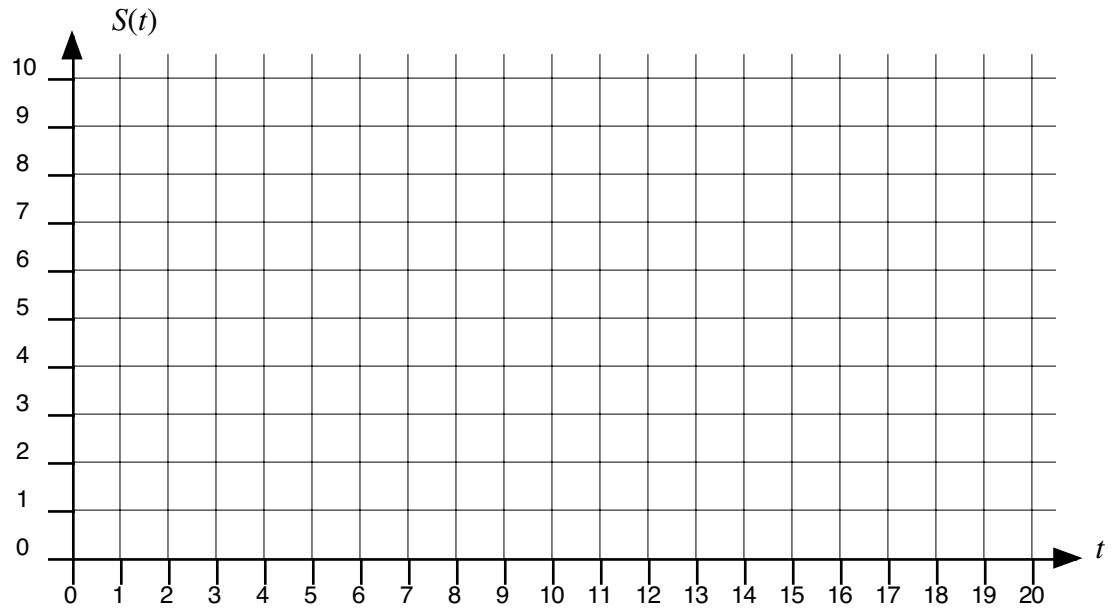
[5] Source: Ministry of Agriculture, UK. http://www.ruralni.gov.uk/livestock/beef/technology_projects/previous_projects/continental_bulls.htm

[6] Source: Tacon, A. J. G. 1996. *International Aquafeed Directory and Buyer's Guide*. Middlesex UK: Turret Research Associates International.

[7] Most of the fishmeal and oil used in salmon farming currently comes from South America. Source: Naylor, R. L., R. J. Goldburg, H. Mooney, M. Beveridge, J. Clay, C. Folke, N. Kautsky, J. Lubchenco, J. Primevera and M. Williams. 1998. Nature's subsidies to shrimp and salmon farming. *Science*, 282: 883-884.

[8] These equations and function values were created from data contained in:

- Food and Agriculture Organization of the United Nations. 1998. *World Aquaculture Production Statistics*. Rome Italy: United Nations.
- Rosenberry, B. 1996. *World Shrimp Farming*. San Diego CA: Shrimp News International, Inc.



(b) In 1982 ($t = 0$) the total world shrimp production was about 100,000 metric tons ($S(0) = 1$). Use this information together with the slope field that you have drawn to sketch an approximate graph showing world shrimp production as a function of time.

(c) In the year 2002 ($t = 20$) what (approximately) was the total world production of farmed shrimp? Express your answer in metric tons.

(d) How many metric tons of non-commercial fish must be caught and rendered into fishmeal and fish oil in order to supply the farmed shrimp with adequate nutrition? Express your answer in units of metric tons.

2. Let h be the function whose graph is shown in Figure 3 below. The domain of the function h is the interval [0, 4].

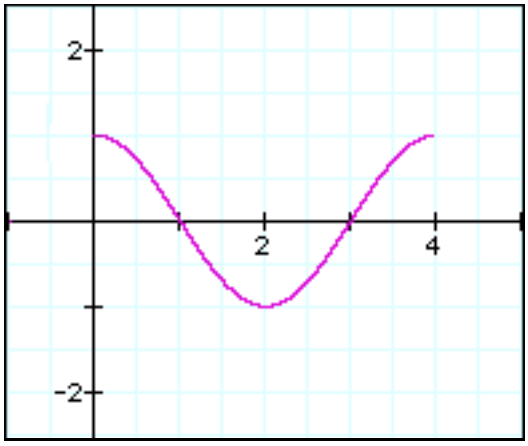


Figure 3: Graph of y = h(x).

(a) Over what intervals is h an increasing function? Over what intervals is h a decreasing function?

(b) Over what intervals is h a concave up function? Over what intervals is h a concave down function?

(c) Where are the zeros (i.e. the x-intercepts) of the function h located?

(d) A new function p is defined by the equation:

$$p(x) = 2 \cdot h(-x) + 2.$$

What is the domain and range of the function p?

(e) Sketch an accurate graph showing $y = p(x)$. Where are the zeros of the function p located?

3. Many food and wine magazines rank the quality of wine on a scale of 1 to 100, 100 being the finest. The scores that a developing Californian vineyard achieved for its Chardonnay are recorded in Table 1.

Vintage	1977	1982	1983	1985	1990	1993
Score	72	78	82	90	85	90

Table 1

(a) Plot a graph showing the score for the Chardonnay versus the year. What kind of function (linear, exponential or power) would do the best job of representing this relationship?

(b) Find an equation for the score as a function of time. What score would you predict for the vineyard's 1980 Chardonnay ?

(c) The vineyard started to produce Chardonnay in 1969. According to your equation, what score did their first vintage receive? How reliable do you think this figure is, based on the appearance of your plot from Part (a)?

(d) Vineyards that consistently produce wines which consistently score 92 or above are regarded as elite. In a speech to wine growers, the owner of the vineyard confidently stated that by 1995, this vineyard would rank amongst the elite. Do you think these claims are valid or not?

4. The world championship of hot dog eating is held on the Fourth of July each year in New York city. In recent years, the event has been dominated by competitors from Japan who typically weigh only about 100 lb. In 2000, Japanese competitors finished first, second and third. Third place went to Ms. Takako Akasaka (the only woman in the competition) who set a world record for female hot dog eating by consuming 22.25 hot dogs in 12 minutes. Let $f(t)$ represent the total number of hot dogs eaten by Ms. Akasaka after t minutes. Some values of $f(t)$ are given in Table 2 below.

t	0	2	4	5	8	10	12
f(t)	0	5	11	14	18	20	22.5

Table 2

Use the information about $f(t)$ to answer the following questions.

(a) Estimate $f'(2)$ as accurately as possible. What does your answer mean in practical terms?

(b) The eating competition lasted for twelve minutes. If $f(3) = 8$, and $f'(3) = 2.3$, estimate the number of hot dogs that Ms. Akasaka had eaten after four and a half minutes.

(c) Is the approximation that you have calculated in part (b) probably greater than or probably less than the actual number of hot dogs eaten by Ms. Akasaka in the first four and a half minutes of the contest? Be careful to supply a mathematical justification for your conclusion.

(d) Explain the meaning of the mathematical symbols: $f^{-1}(10)$ in practical terms.

5.(a) Draw an accurate graph of the inverse of : $y = x^3 - x^2 - 6$.

(b) Either find a formula for the inverse of the function :

$$q(x) = ln\,(x + 3) - ln\,(x - 5)$$

or give an argument why no such formula exists.

6. The graph shown below in Figure 4 is the graph of $y = f(x) = F'(x)$.

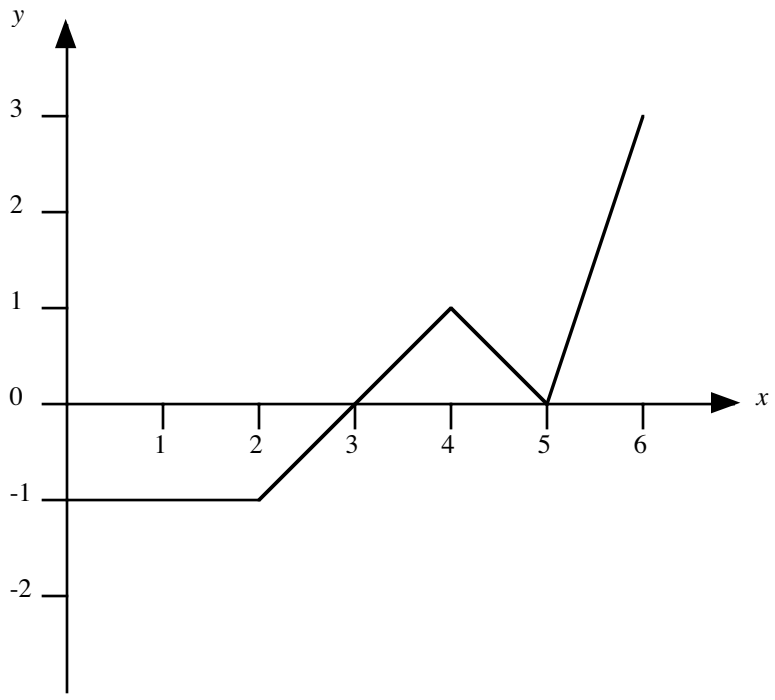


Figure 4: Graph of $y = f(x)$.

(a) Locate and classify the critical points of $F(x)$.

(b) Locate the x-coordinates of the global maximum and minimum of $F(x)$ on the interval [0, 6].

(c) Locate any inflection points of $F(x)$. How do you know where the inflection points of $F(x)$ will occur?

(d) Suppose that the one other thing that you are told is that $F(2) = 14$. Sketch a graph of $y = F(x)$. Label the points that you have found in parts (a)-(d) of this problem.

7. The points that satisfy equation,

$$x^2 - xy + y^2 = 7.$$

form an ellipse in the x-y plane.

(a) Verify that the point (1, 3) is on the ellipse.

(b) Find an equation for $\frac{dy}{dx}$.

(c) Find the slope of the tangent line to the ellipse at the point (1, 3).

(d) Are there any points on the ellipse where the tangent line is horizontal? If so, find the x-coordinates of these points. If not, explain why it is impossible for the tangent line to be horizontal.

8. In this problem, the functions f, g and h will always refer to:

- $f(x) = x^2 - 4$
- $g(x) = x^2 + 4$
- $h(x) = x + 5$.

Using these three functions as building blocks, create equations for functions that have the properties describe below.

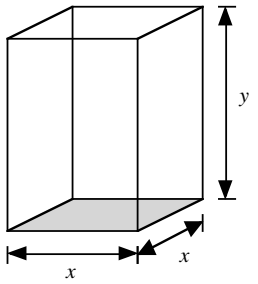
(a) The function is defined for all real numbers. The function has a horizontal asymptote at $y = 0$ and an x-intercept at $x = -5$.

(b) The function has x-intercepts at $x = -5$, $x = -2$ and $x = 2$. The function does not have any horizontal or vertical asymptotes.

(c) The function has no x-intercepts. The function has a horizontal asymptote at $y = 0$. The function has no vertical asymptotes.

9. The New England Aquarium is planning a new exhibit to house mantis shrimp. Mantis shrimp (*Squilla empusa*) are small crustaceans with very powerful claws. Tanks designed to hold mantis shrimp need to be specially constructed to withstand blows from the shrimps' powerful claws. (Several exhibits of mantis shrimp have been destroyed - by the shrimp in the exhibit - when the shrimp struck the bottom of the tank and broke the glass of the tank.)

The New England Aquarium are planning to build a tank that has glass sides and a slate bottom (see diagram below). The tank will have a square bottom, no top, and should have a volume of 100 cubic feet.



(a) Slate costs \$10 per square foot and glass costs \$2 per square foot. Find a formula for the cost of the aquarium.

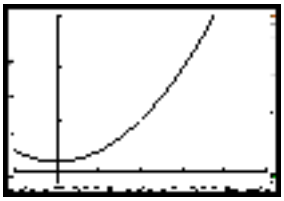
(b) If the New England Aquarium wants to build the cheapest tank possible, what dimensions should the tank have?

(c) How do you know that the dimensions that you gave in part (b) will give the cheapest tank possible?

10. In this problem, the function f will always refer to the function defined by the equation given below.

$$f(x) = \frac{x^3 - 2x^2 + x - 2}{x - 2}.$$

(a) If you were to graph $y = f(x)$ on a graphing calculator, then (depending on what size viewing window you chose) you would get a display that was something like the one shown below.



What is unusual about the graph that the calculator has produced? This is not a calculator malfunction. What has caused the unusual feature, and where (i.e. what x-value) is the unusual feature located?

(b) If you evaluate $f(x)$ at several x-values just to the left of the unusual feature, what y-value do you approach? How could you express the situation using limit notation?

(c) If you evaluate $f(x)$ at several x-values just to the right of the unusual feature, what y-value do you approach? How could you express the situation using limit notation?

(d) The equation defining the function f can also be expressed in the form shown below:

$$f(x) = \frac{(x^2 + 1)(x - 2)}{x - 2}.$$

Use this form of the function f to explain the y-values that you calculated in Parts (b) and (c) of this problem.