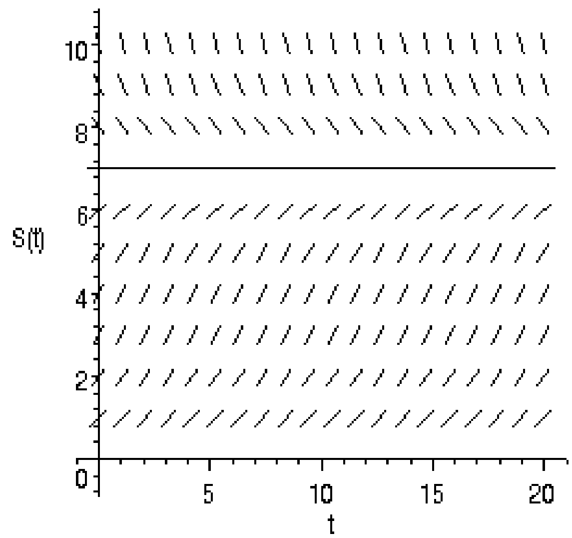
Math Xa **Fall 2002**

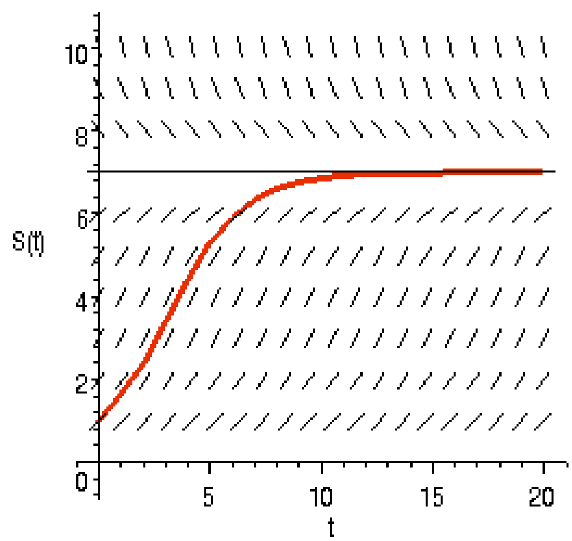
Solutions: Final Exam – Set #4

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The slope field for the differential equation is shown below.



1.(b) The slope field with a plausible graph superimposed on it is shown below. Note that the graph starts at the point (0, 1) because the only definite information that you are given about this function is that $S(0) = 1$.



1.(c) Reading from the graph in Part (b), $S(20) \approx 7$. So, the total world shrimp production this year was approximately 700,000 metric tons.

1.(d) The FCR for a tiger shrimp is 4:1. This means that to produce 700,000 metric tons of shrimp, about 2,800,000 metric tons of feed are required. In the case of shrimp farming, about 33% of this is fishmeal and fish oils, so at least 0.33×2800000 = 924,000 metric tons of non-commercial fish species would have to be caught in order to provide enough animal protein to feed all of the farmed shrimp.

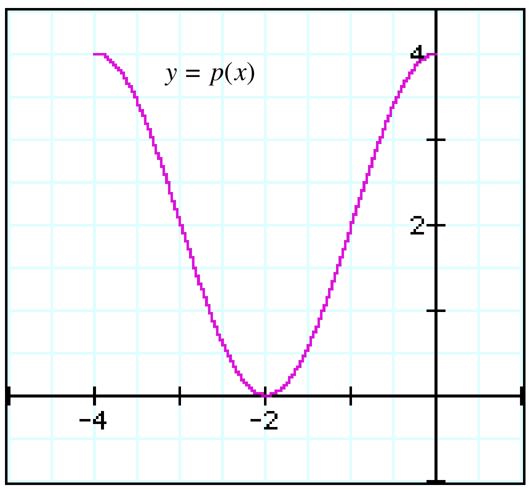
2.(a) The function h is increasing on the interval (2, 4). The function h is decreasing on the interval (0, 2).

2.(b) The function h is concave up on the interval (1, 3). The function h is concave down on the intervals (0, 1) and (3, 4).

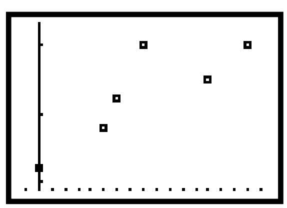
2.(c) The zeros of h are located at $x = 1$ and $x = 3$.

2.(d) The domain of the new function p is the interval [−4, 0]. The range of the new function p is the interval [0, 4].

2.(e) The graph of $y = p(x)$ is shown in the diagram below. The only zero of the function p occurs at $x = -2$.



3.(a) The plot of the data will resemble the graph shown below.



Based on the appearance of this plot, I would suspect that either a linear or possibly an exponential function to represent the relationship between score and time. However, I would favor a linear equation over an exponential equation because there is no sign of a strongly concave up pattern in the plot. A concave up pattern could suggest that an exponential equation would be more appropriate.

3.(b) Using linear regression on a calculator with T = years since 1977 as the independent variable and S = score as the dependent variable gives the equation:

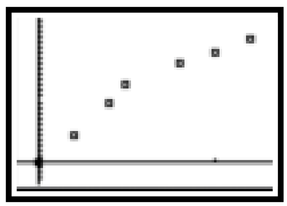
$$S = 1.03 \cdot T + 74.59.$$

For the 1980 score, you would plug $T = 3$ into this equation, yielding: $S = 77.68$.

3.(c) The year 1969 would correspond to $T = -8$. Plugging this into the equation for S gives: $S = 66.35$. Predicting the value of a function outside of the given data set is always somewhat risky. For example, vineyards usually produce absolutely terrible wine at first because their grape vines are not mature, and are not producing wonderful fruit. Likewise, if the wine-makers are novices at the beginning then they will likely make a lot of mistakes that could also diminish the quality of the wine. I would say that the score of 66.35 probably represents a "best case scenario" for the quality of wine that the vineyard produced during its first year.

3.(d) If the trends shown in Table 1 continue, then this is a not unreasonable statement, as the prediction of the equation for the year 1995 is $S = 93.13$, and it continues to rise after that. However, wine making is subject to a lot of unpredictable factors such as the climate, the weather, disease, the expertise of the vintners, etc. If any of these factors fluctuated a lot during the next few years then the chairman's prediction might be overly optimistic.

4.(a) Here you don't have an equation for the function $f(t)$. One possibility is to try to fit a function to the data in Table 2. If you graph the data in Table 2, you will get a plot resembling the one shown below.



This graph doesn't look much like any of the functions that we are accustomed to - it looks more like a piece-wise defined function consisting of two linear functions. A collection of functions that you could use to describe this relationship is:

Time interval	f(t)
0 ≤ t < 5	(14/5)*t
5 ≤ t ≤ 12	5 + (8.5/7)*t

Based on this, $f'(2) = 14/5$. Remember, though, that this is a reasonable approximation for the data that we have - in reality, if we had more information about the value of the function, we might find the that value of $f'(2)$ could be quite different from 14/5.

4.(b) Given this information, the equation of the tangent line based at t=3 is given by:

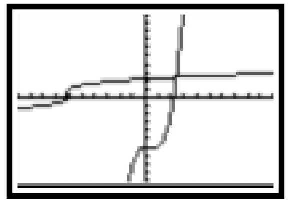
$$y = 2.3 \cdot (t - 3) + 8.$$

Using this equation to estimate the number of hot dogs eaten when $t = 4.5$: $y = 11.45$.

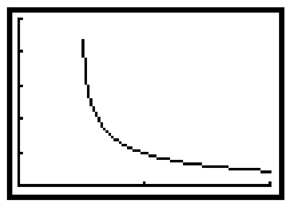
4.(c) According to Table 38.1, Ms. Akasaka had eaten 11 hot dogs at t=4 and 14 at t=5. My guess is that 11.45 hot dogs eaten at t=4.5 is an underestimate. According to Table 2, Ms. Akasaka ate 3 hot dogs between t=4 and t=5. At t=4.5, I'd expect to her to have eaten at least one of those three hot dogs, so I'd expect the actual number eaten to be 12 or higher when t=4.5.

4.(d) $f^{-1}(10)$ is the number of minutes that have elapsed in the competition when Ms. Akasaka finishes her tenth hot dog.

5.(a) A plot showing both the function and its inverse is shown below. The function is the graph that goes up and down, whereas the inverse is the graph that goes left to right. The graph of the inverse was obtained by reflecting the graph of the function across the line $y = x$.



5.(b) Graphing the function $q(x) = \ln(x + 3) - \ln(x - 5)$ gives a graph like the one shown below. (The window size is xmin=0, xmax=20, ymin=0 and ymax=5.)



This graph appears to pass the Horizontal Line Test, so you would expect $q(x)$ to have an inverse. One way to find an equation for the inverse is to solve the equation

$$y = \ln(x + 3) - \ln(x - 5)$$

to make x the subject of the equation. Doing this:

$$y = \ln\left(\frac{x+3}{x-5}\right)$$

$$e^y = \frac{x+3}{x-5}$$

$$(x - 5)\cdot e^y = x + 3$$

$$x\cdot e^y - x = 5\cdot e^y + 3$$

$$x = \frac{5\cdot e^y + 3}{e^y - 1}$$

6.(a) The critical points of $F(x)$ are the places where $f(x) = 0$. These occur at $x = 3$ and $x = 5$. To classify these:

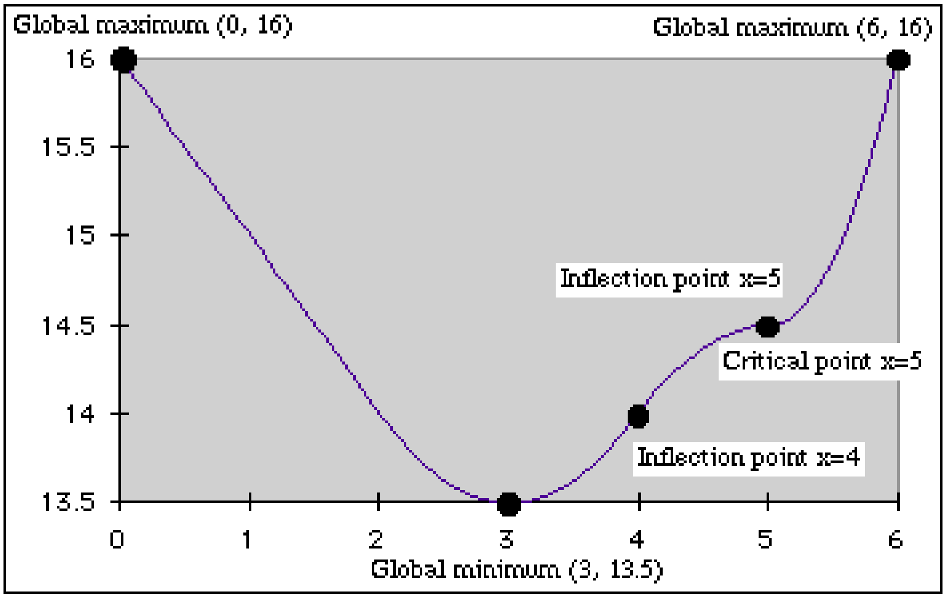
- $x = 3$: The graph of $f(x)$ is negative just before x=3 and positive just after x=3, so by the First Derivative test, x=3 is a local minimum.

- $x = 5$: The graph of $f(x)$ is positive just to the left and just to the right of x=5. This point is neither a maximum nor a minimum.

6.(b) The global minimum is located at x=3. The global maximum occurs at x=0 and x=6.

6.(c) Inflection points are where the graph of $F(x)$ changes concavity. On Figure 4, this is indicated by the graph changing from increasing to decreasing or vice versa. Looking at Figure 4, you can see that this happens at x=4 and x=5. So, $F(x)$ has points of inflection at x=4 and x=5.

6.(d) The graph of $y = F(x)$ with $F(2)=14$ is shown below.



7.(a) To verify that the point (1, 3) lies on the ellipse, all that you need to do is substitute x=1 and y=3 into the equation and make sure that you get '7' as the result.

7.(b) Differentiating the equation term by term gives:

$$2x - y - x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0.$$

Re-arranging to get everything involving dy/dx on one side of the equation and everything that does not involve the derivative on the other side of the equation gives:

$$-x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = y - 2x.$$

Factoring out the dy/dx and then making dy/dx the subject of the equation gives:

$$\frac{dy}{dx} = \frac{y-2x}{-x+2y}.$$

7.(c) To find the equation of the tangent line, you need to know the slope and the intercept. To find the slope, substitute $x = 1$ and $y = 3$ into the equation for dy/dx. This gives slope = m = 0.2. To find the intercept, substitute $x = 1$, $y = 3$ and $m = 0.2$ into the equation for a linear equation:

$$y = mx + b.$$

This gives $b = 2.8$, and the equation of the tangent line: $y = 0.2 \cdot x + 2.8$.

7.(d) In order for the tangent line to the ellipse to be horizontal, the derivative dy/dx must be equal to zero. From the equation for the derivative, this requires that: $y = 2x$. Substituting this into the original equation for the ellipse gives:

$$x^2 - x \cdot (2x) + (2x)^2 = 7$$

Simplifying this equation gives: $3x^2 = 7$.

Solving this equation for x: $x = \pm\sqrt{(7/3)}$.

8.(a) $y = \dfrac{x+5}{x^2+4}$.

8.(b) $y = (x+5)\cdot(x^2-4)$ or $y = \dfrac{(x+5)\cdot(x^2-4)}{x^2+4}$.

8.(c) $y = \dfrac{1}{x^2+4}$.

9.(a) The cost is equal to: $10x^2 + 8xy$. This is too many variables to differentiate, so you need to do something to eliminate one of the variables. You are told that the volume is 100, so $x^2y = 100$, giving: $y = 100/x^2$. Substituting this into the equation for the cost gives that the cost, $C(x)$ is given by:

$$C(x) = 10x^2 + \frac{800}{x}.$$

9.(b) The dimensions of the tank that minimizes cost are: x = (40)^(1/3) and y=100/(40)^(2/3).

9.(c) The second derivative of the cost function is: C"(x) = 20 + 1600/x^3. This is positive when x>0, and in particular $C''(40^{1/3}) > 0$. A positive second derivative means that the critical point is a local minimum.

10.(a) The unusual feature is the fact that the graph produced by the calculator has a missing point - that is, the curve is not a continuous curve but instead appears to have a little gap in it. This is caused by the denominator $(x - 2)$. In particular, when $x = 2$, the denominator will be equal to zero and the function f is not defined. Therefore, the gap in the graph on the calculator screen should be located at $x = 2$.

10.(b) If you try a few values of x that are slightly less than $x = 2$ in the function $f(x)$, then the y-values seem to get closer and closer to 5 the nearer that x gets to 2.

10.(c) If you try a few values of x that are slightly greater than $x = 2$ in the function $f(x)$, then the y-values seem to get closer and closer to 5 the nearer that x gets to 2.

10.(d) Think about what each of the brackets in the factored version of $f(x)$ is doing when x is very close to 2.

$$f(x) = \frac{(x^2+1)(x-2)}{x-2} = (x^2+1)\cdot\frac{x-2}{x-2} \approx (5)\cdot 1.$$