Math Xa **Fall 2002**

Practice Problems: Final Exam – Set #5

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.

2. The test will include twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test.

5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

7. Good sources of help:

   - Section leaders’ office hours (posted on Xa web site).
   - Math Question Center (during the reading period).
   - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.

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1. A New York judge has found a company guilty of fraud. The judge has noted that the company executives have particularly bad attitudes, and will probably not pay their fines on time. The judge decides to offer the company a choice of punishments[1].

Scheme A: The company has to pay $10,000,000 plus $1,000,000 for each additional day that payment is not received.

Scheme B: The fine begins at $1. The fine doubles in size each day until it is paid.

(a) Write down an equation that gives the size of the fine as a function of time if scheme A is selected.

(b) Write down an equation that describes the size of the fine if scheme B is selected.

(c) Over what period of time would it be most economical for the company to pay the fine from Scheme B? When does Scheme B become the most expensive fine?

(d) The Gross Domestic Product[2] of the US in 1998 was $8,511,000,000,000. How many days will it take for the fine in Scheme B to equal this figure?

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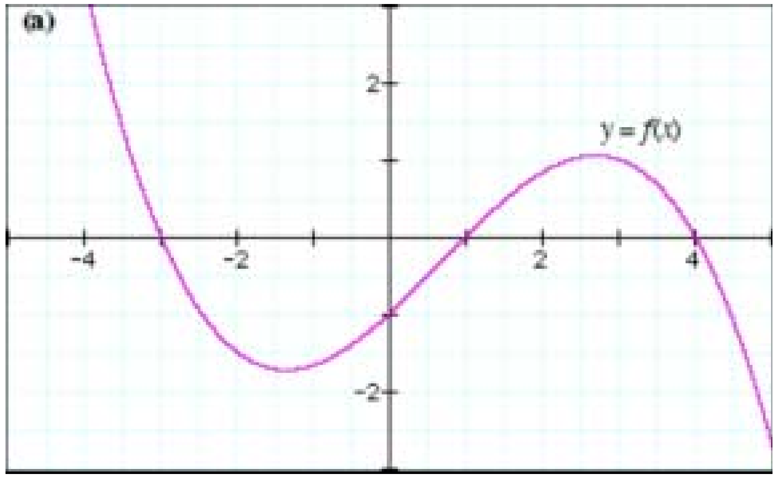
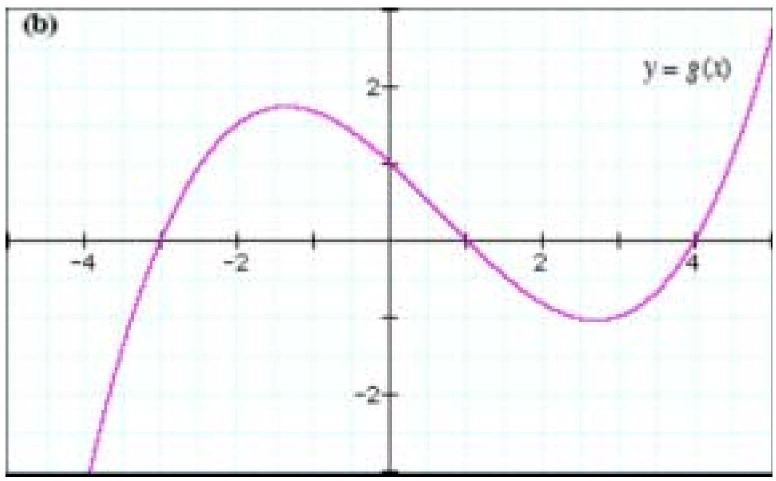
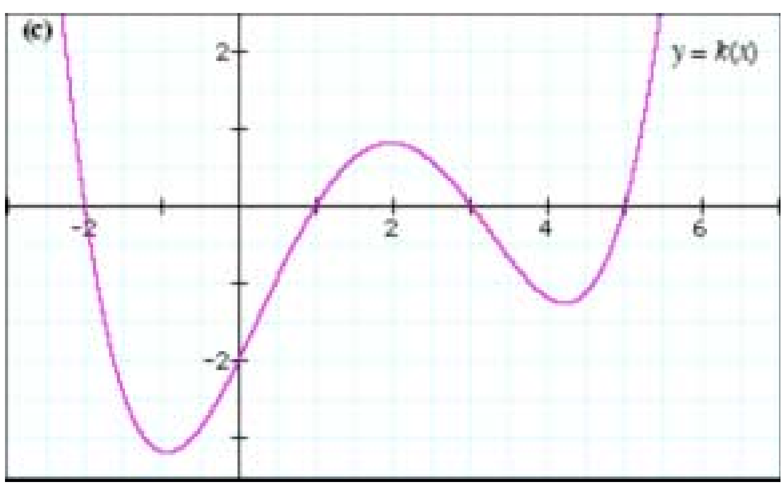
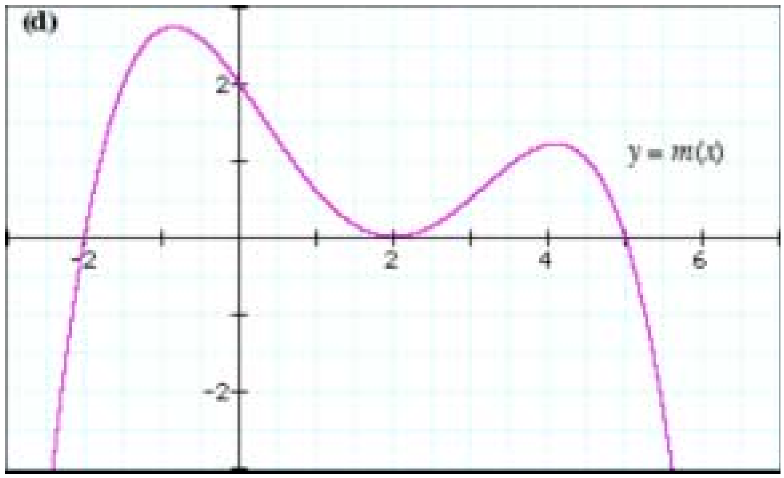
[1] Believe it or not, this problem is actually based on a true story!
[2] Source: US Bureau of Economic Analysis, *Survey of Current Business*, May 1999.

2. The graphs given below show the graphs of **polynomial** functions. For each graph, find the equation for the polynomial function.



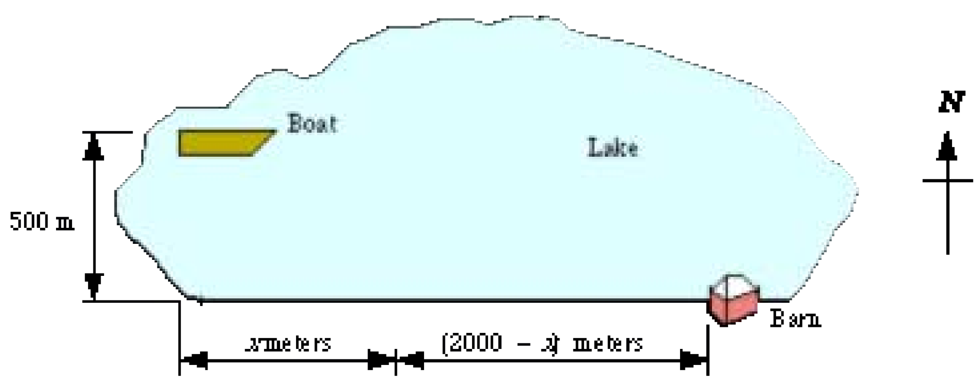






3. Under some conditions, it is easier for a bird to fly over land than it is for the bird to fly over water. This is because the temperature of a large body of water (like a lake or the ocean) is much more stable than the temperature of the land. Land tends to heat up more than water does, creating thermal updrafts that make it easier for the birds to stay aloft. One reference[3] suggests that when flying over water, homing pigeons have to expend 30 joules for every one meter that they fly. The same pigeons only have to expend 10 joules for every one meter that they fly over land.

The diagram below shows the set-up for an annual race held by pigeon enthusiasts. The pigeons are released from a boat in the middle of a lake, and the finish line is a barn on the edge of the lake. The pigeons fly at the same speed no matter whether they fly over water or over land.



When released from the boat, the pigeons normally fly in a diagonal line towards the shore. When they reach the shore, the pigeons normally fly along the shore until they reach the barn, where they land.

(a) Find an equation that describes the amount of energy expended by a pigeon during the race. (The variable in your equation should be *x*.)

(b) Generally speaking, most animals try to conserve energy when possible. Where should the pigeon come ashore in order to expend the least amount of energy during the race?

(c) Recall that pigeons fly at the same speed regardless of whether they are flying over land or water. This is about 5 meters per second. Find an equation that gives the amount of time that a pigeon needs to complete the race. (The variable in your equation should be *x*.)

[3] Deborah Hughes-Hallet, Andrew Gleason, et al. "Calculus." New York: John Wiley and Sons, 1994.

(d) Where should the pigeon come ashore in order to complete the race in the least amount of time?

4. Doug is pouring water into a large spherical tank at a constant rate. Let $H(t)$ be the height of water in the tank at time t and let $V(t)$ be the volume of water in the tank at time t.

(a) Is $\frac{dH}{dt}$ positive, negative or zero when the tank is one quarter full?

(b) Is $\frac{d^2H}{dt^2}$ positive, negative or zero when the tank is one quarter full?

(c) Is $\frac{dV}{dt}$ positive, negative or zero when the tank is one quarter full?

(d) Is $\frac{d^2V}{dt^2}$ positive, negative or zero when the tank is one quarter full?

5. The table below gives the number of people (in thousands) receiving Medicaid, and the payments made to Medicaid vendors (in millions of dollars) between 1975 and 1997[4].

Year	Number of recipients (thousands)	Vendor payments (millions of dollars)
1975	3615	4358
1981	3367	9926
1985	3061	14096
1990	3202	21508
1995	4119	36527
1996	4285	36947
1997	3954	37721

(a) Plot a graph showing the average expenditure of Medicaid per recipient of Medicaid between 1975 and 1997. What kind of function would do a good job of representing the average expenditure on Medicaid per recipient as a function of time?

(b) Find an equation for average expenditure of Medicaid as a function of time.

(c) Plot a graph showing the number of recipients of Medicaid versus year for 1975 and 1997. What kind of function would do a good job of representing the number of recipients of Medicaid as a function of time?

(d) Find an equation for number of recipients of Medicaid as a function of time.

(e) How could you combine the equations that you found in parts (b) and (d) of this problem to create an equation that would give the total expenditure on Medicaid as a function of time?

6. In this problem the function $g(x)$ will always refer to the function defined by the equation:

$$g(x) = (x-2)^2(x-6)^2 + 1$$

(a) Locate all of the points where $g'(x) = 0$.

(b) Classify the points that you have found in Part (a) - i.e. are the points "hill-tops," "valley bottoms" or neither? Be careful to provide convincing evidence for your conclusions.

[4] Source: Health Care Financing Administration, *2082 Report* (1999).

(c) Find the locations of the points where the concavity of $g(x)$ changes.

(d) Sketch an accurate graph of $y = g(x)$.

7. The curve defined by the equation:

$$x^3 + y^3 = 6x \cdot y$$

is known as the "Folium of Descartes."

(a) Show that the point $(x, y) = (3, 3)$ lies on the folium. Find the coordinates of one more point that lies on this curve.

(b) Find a formula for the derivative of y with respect to x, that is, a formula for: $\frac{dy}{dx}$.

(c) Find an equation for the tangent line to the folium, that touches the folium at the point (3, 3).

(d) How could you use the derivative, $\frac{dy}{dx}$, to locate places on the folium where the tangent line is horizontal?

(e) Find the coordinates of one point on the folium where the tangent line is horizontal.

8. In this problem, f and g are differentiable functions. All that you can assume about them is the information given in the table below.

f(1) = 2	g(1) = 4	$f'(1) = 3$	$g'(1) = 7$
f(2) = 6	g(2) = −1	$f'(2) = 8$	$g'(2) = 15$
f(3) = 0	g(3) = 1	$f'(3) = 6$	$g'(3) = 17$
f(4) = 8	g(4) = −5	$f'(4) = 9$	$g'(4) = 21$

Use the information given in the table to evaluate the following:

(a) $a'(2)$ where $a(x) = f(x) \cdot g(x)$.

(b) $b'(1)$ where $b(x) = \dfrac{f(x)}{g(x)}$.

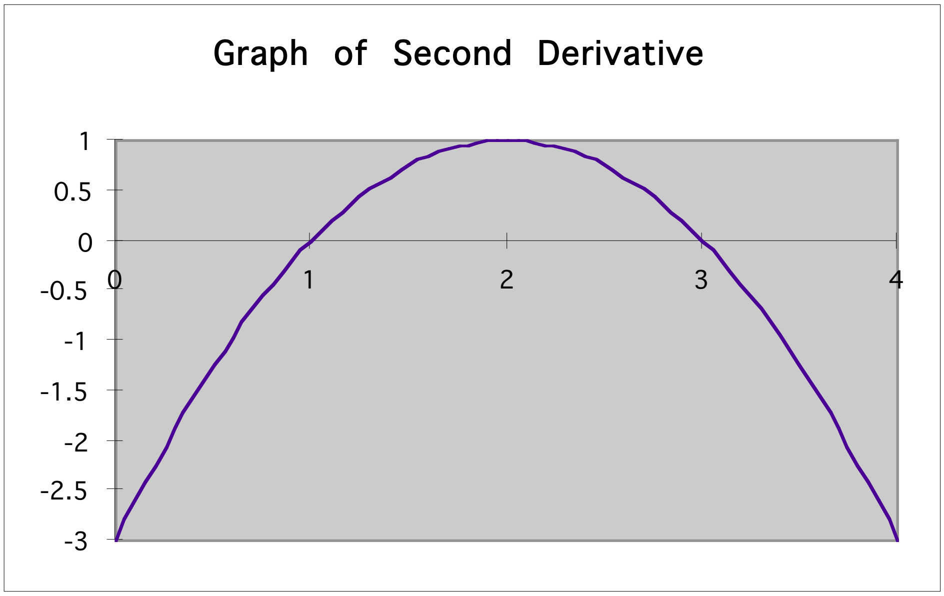
(c) $c'(3)$ where $c(x) = f(x) + g(x)$.

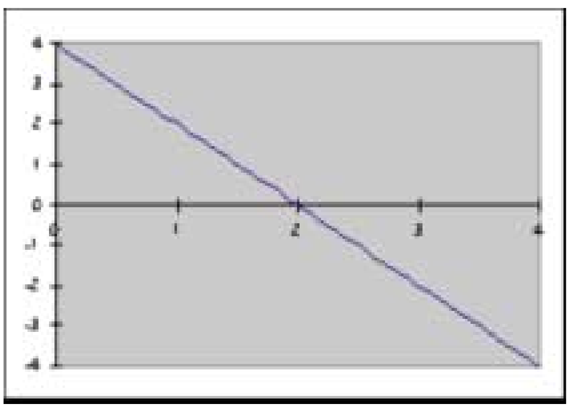
(d) $d'(1)$ where $d(x) = f(g(x))$.

(e) $k'(1)$ where $k(x) = g(f(x))$.

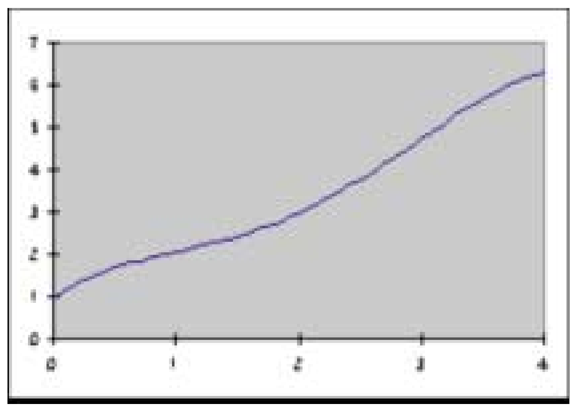
(f) $k'(2)$ where $k(x) = g(f(x))$.

9. The graph given below is the graph of the *second derivative* of a function. Explain which of the graphs (a, b, c or d) given below could be the graph of the original function.

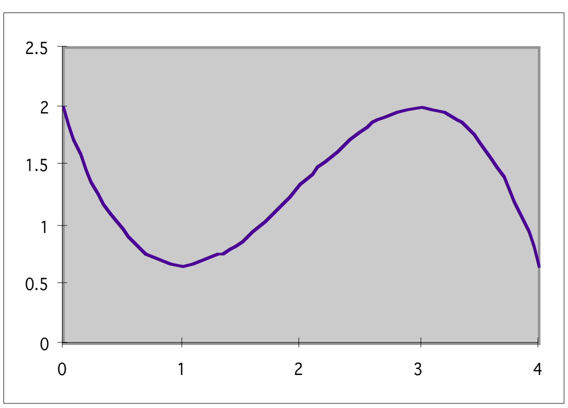




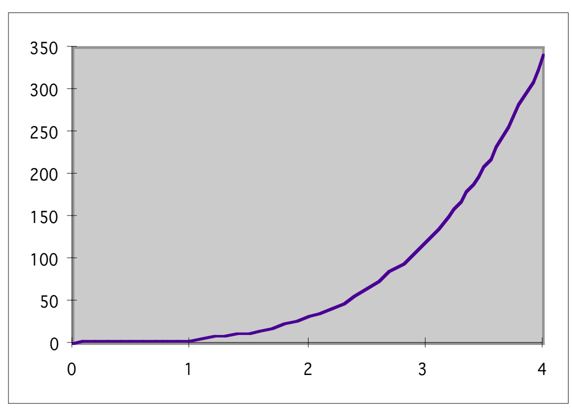
Graph A.



Graph B.



Graph C.



Graph D.

10. In Math Xa, you learned about Newton's Law of Cooling which could be expressed as an equation for the derivative of temperature with respect to time. In this problem, at time $t = 0$ a boiled egg is dropped in a bowl of water to cool. At $t = 0$ the temperature of the egg is 98ºC. The temperature of the water in the bowl is 18ºC. (You can assume that the bowl has enough water in it so that the temperature of the water does not change very much.)

(a) As time goes by (i.e. as $t \to \infty$) what value would expect the temperature of the egg to approach?

(b) Sketch a graph showing the temperature of the egg versus time.

One of the things that you will learn to do in Math Xb is find a formula to represent the temperature of an object that is cooling, by working backwards from Newton's Law. The form of the equation that you will learn to obtain for temperature, $T(t)$, as a function of time, t, is:

$$T(t) = C + A \cdot e^{k \cdot t}.$$

where C, A and k are constants, and e is the special number that you encountered in Math Xa.

(c) Based on your answer to Part (b), do you expect k to be a positive number or a negative number?

(d) Using the formula for $T(t)$ given above, what is: $\lim_{t \to \infty} T(t)$?

(e) Based on your answer to Part (a), what is the value of the constant C?

(f) In the description of the problem, you were told that when $t = 0$, the temperature of the egg was 98ºC. Use this information to find the value of the constant A.

(g) After 5 minutes in the water, the temperature of the egg had fallen to 38ºC. Use this information to find the value of the constant k.

(h) How long will it take for the temperature of the egg to fall to 20ºC? How quickly is the temperature falling at the instant of time when the temperature reaches 20ºC?