Math Xa **Fall 2002**

Solutions: Final Exam – Set #5

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) Let $A(T)$ = fine (in $) after T days if scheme 'A' is selected. Then:

$$A(T) = 10000000 + 1000000*T.$$

1.(b) Let $B(T)$ = fine (in $) after T days is scheme 'B' is selected. Then:

$$B(T) = 2^T.$$

1.(c) Scheme 'B' will be the cheaper alternative up until the point when $B(T) = A(T)$. Exactly when this happens can be determined using the graphing calculator in much the same way as the previous problem. In this case, the intersection of the two graphs occurs at: $T = 25.063$. So, up until the 25th day, scheme 'B' is the cheapest. On day 25, the fines from schemes 'A' and 'B' are roughly equal, and after day 25, scheme 'B' is more expensive.

1.(d) The equation that you have to solve (for T) is: $8{,}511{,}000{,}000{,}000 = 2^T$. Using logarithms to solve this gives:

$$T = \log(8{,}511{,}000{,}000{,}000)/\log(2) = 42.95.$$

So, after about a month and a half, the fine will have grown to the size of the GDP of the entire United States.

2.(a) $f(x) = (-1/12)\cdot(x + 3)\cdot(x - 1)\cdot(x - 4)$

2.(b) $f(x) = (1/12)\cdot(x + 3)\cdot(x - 1)\cdot(x - 4)$

2.(c) $f(x) = (1/15)\cdot(x + 2)\cdot(x - 1)\cdot(x - 5)$

2.(d) $f(x) = (-1/20)\cdot(x + 2)\cdot(x - 2)^2\cdot(x - 5)$

3.(a) The total amount of energy expended by the pigeon, $E(x)$, is given by:

$$E(x) = 30\cdot[500^2 + x^2]^{1/2} + 10\cdot(2000 - x).$$

3.(b) If you take the derivative of the function $E'(x)$ you get:

$$E'(x) = \frac{30x}{\sqrt{500^2 + x^2}} - 10.$$

If you set this derivative equal to zero and then solve for x then you get: $x = 500/\sqrt{8}$.
To check that this is a minimum, you can evaluate the derivative at an x-value slightly smaller than this and at an x-value slightly larger than this. For example:

x	176.7	176.8
Derivative	–0.00386	0.00117

As the derivative is negative to the left of the critical point and positive to the right of the critical point, the critical point is a local minimum.

3.(c) Note that speed = distance/time, so that time = distance/speed. Therefore, the amount of time, $T(x)$, in seconds that the pigeon needs to complete the race will be given by:

$$T(x) = \frac{\sqrt{500^2 + x^2} + 2000 - x}{5}.$$

3.(d) As the speed is the same over land and water, the quickest time will be achieved if the pigeon minimizes the amount of distance covered. The pigeon can do this by flying directly from the boat to the barn in a straight line. Although this expends lots of energy, it gives the minimum possible time for the pigeon to complete the race.

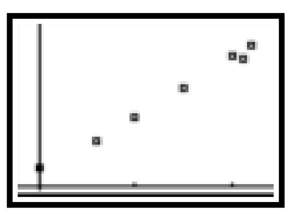
4.(a) The derivative of height with respect to time, $\frac{dH}{dt}$, is positive when the tank is one-quarter full. This is because as time passes, the height of water in the tank increases.

4.(b) The second derivative of height with respect to time, $\frac{d^2H}{dt^2}$, is negative when the tank is one-quarter full. This is because the tank gets wider as you get higher in it. Therefore, as you pour water into the tank, the height of water will increase but not at quite as fast a rate. This means that a graph of height versus time will be a concave down graph, and when the graph of the original function is concave down, the second derivative of the function is negative.

4.(c) Water is being poured into the tank, so the volume of water in the tank should be increasing. Therefore, the derivative of volume with respect to time (i.e. $\frac{dV}{dt}$) is positive.

4.(d) The second derivative is equal to zero. This is because on one hand the second derivative is the derivative of the derivative. On the other hand, water is added to the tank at a constant rate, so $\frac{dV}{dt}$ is a constant. The derivative of a constant is equal to zero.

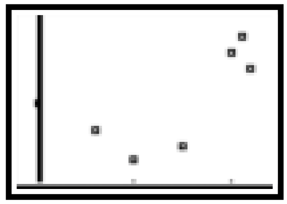
5.(a) The plot is shown below. To find the average expenditure per recipient, the Medicaid payments were divided by the number of Medicaid recipients for each year.



The plot shown above shows that the average cost of Medicaid (per recipient) has been steadily rising since 1975. The costs seem to be rising in a steady fashion, with the data points lying in an almost perfectly straight line. Based on the fact that the data points lie in an almost perfectly straight line, a linear function will probably do a very good job of representing the relationship between the average cost and time.

5.(b) If T = years since 1975, and A = average Medicaid payments (in thousands of dollars), then linear regression on a calculator gives the equation: $A = 0.38 \cdot T + 0.93782$.

5.(c) A plot of the number of recipients of Medicaid versus time is given below.



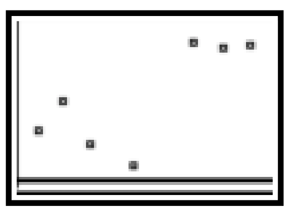
Based on the appearance of the plot, a quadratic function could do a reasonable job or representing this trend in the data.

5.(d) Using T as the independent variable and N = number of recipients of Medicaid (in thousands) be the dependent variable, quadratic regression on a calculator gives:

$$N = 6.43 \cdot T^2 - 115.67 \cdot T + 3662.18.$$

5.(e) The total expenditure on Medicaid is the average expenditure per recipient times the total number of recipients. If T is the variable defined in part (b) and E is total expenditure on Medicaid (in millions of dollars) then: $E = A \cdot N = (0.38 \cdot T + 0.93782) \cdot (6.43 \cdot T^2 - 115.67 \cdot T + 3662.18)$.

5.(f) A plot of the expenditure on Medicaid (in millions of dollars) versus the number of recipients (in thousands) of Medicaid is given below.



Based on this plot, it is not at all easy to say what kind of function would do a good job of representing this data. A linear function is perhaps the simplest function that you could use, and will probably do as good a job as about any other function that you could come up with (although the data is, of course, very far from the straight line pattern you would like to see for a linear function). If you wanted to do something a bit more complicated, then a quartic function would probably do the best job of creating a function whose graph gets fairly close to most of the data points.

6.(a) Perhaps the simplest way to differentiate $g(x)$ is simply to multiply out all of the brackets and then differentiate term-by-term. This disadvantages of this straight-forward approach are that you firstly have to multiply everything out correctly, and secondly, you are left with a derivative that is not very easy to factor. (It is then quite difficult to find the points where $g'(x) = 0$.)

Adopting a slightly less straight-forward approach using the product rule:

$$g'(x) = \left[\tfrac{d}{dx}\left(x^2 - 4x + 4\right)\right] \cdot (x-6)^2 + (x-2)^2 \cdot \left[\tfrac{d}{dx}\left(x^2 - 12x + 36\right)\right].$$

Differentiating and simplifying gives:

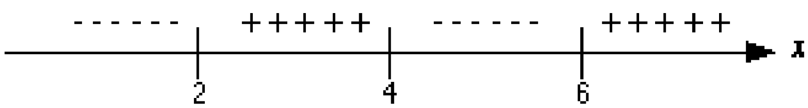
$$g'(x) = 2(x-2)(x-6)^2 + 2(x-2)^2(x-6) = 2(x-2)(x-6)(2x-8).$$

From this, the points where $g'(x) = 0$ are located at $x = 2$, $x = 6$ and $x = 4$. The coordinates of these points are: (2, 1), (6, 1) and (4, 17).

6.(b) There are many ways of classifying the points found in Part (a). One straight-forward way is to examine the sign of $g'(x)$ on either side of the x-value in question.

g'(x) just to the left of the point	g'(x) just to the right of the point	The point is a:
−	+	valley bottom
+	−	hill top
−	−	neither
+	+	neither

The number line for $g'(x)$ is:



Using the number line with the interpretations given in the table above:

- (2, 1) “valley bottom”
- (6, 1) “valley bottom”
- (4, 17) “hill top”

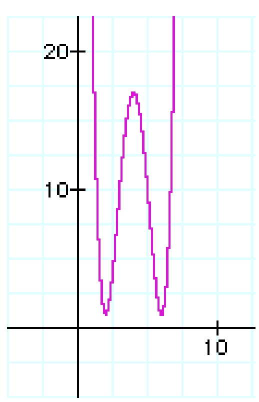
6.(c) The points where the concavity changes are the places where $g'(x)$ stops increasing and starts decreasing, or else places where $g'(x)$ stops decreasing and starts increasing. These are places where the derivative of $g'(x)$ is zero - that is, places where the derivative of the derivative is equal to zero. The derivative of the derivative of $g(x)$ is obtained by differentiating the equation from Part (a).

$$g''(x) = 12x^2 - 96x + 176.$$

Solving the equation: $$12x^2 - 96x + 176 = 0$$

gives $x = 2.845299462$ and $x = 5.154700538$. Therefore, the points where the concavity of the function $g(x)$ changes are the points (2.845299462, 8.111111116) and (5.154700538, 8.111111116).

6.(d) A sketch of $y = g(x)$ is shown below.



7.(a) Plug $x = 3$ and $y = 3$ into the equation and confirm that the left hand side and the right hand side of the equation are equal. There are many points that lie on this curve. Another one is (0, 0).

7.(b) $\dfrac{dy}{dx} = \dfrac{6y - 3x^2}{3y^2 - 6x}$.

7.(c) Use the equation for the derivative to get the slope, m. Plugging $x = 3$ and $y = 3$ into the equation for the derivative gives $m = -1$. Plugging $x = 3$, $y = 3$ and $m = -1$ into the equation for a linear function, $y = mx + b$, gives: $b = 6$. Therefore, the equation of the tangent line is: $y = -x + 6$.

7.(d) You would set the derivative equal to zero, and solve to find an equation relating x and y. You would then use this equation to eliminate one of the variables (x or y) from the equation:

$$x^3 + y^3 = 6x \cdot y$$

that defines the folium. Once you have the equation that defines the folium down to one variable, you would solve to find a numerical value.

7.(e) If you set $dy/dx = 0$, then you get: $y = 0.5x^2$. Using this to replace y in the equation that defined the folium gives:

$$x^3 + \tfrac{1}{8} \cdot x^6 = 3x^3 \Rightarrow x^3 \cdot (2 - \tfrac{1}{8}x^3) = 0.$$

Solving this for x gives either $x = 0$ or $x = (16)^{1/3}$. If $x = 0$ then $y = 0$. If $x = (16)^{1/3}$ then $y = 0.5*(16)^{2/3}$.

8.(a) $a'(2) = 82$.

8.(b) $b'(1) = -1/8$.

8.(c) $c'(3) = 23$.

8.(d) $d'(1) = 63$.

8.(e) $k'(1) = 45$.

8.(f) $k'(2)$ cannot be evaluated with the information you are given. In order to evaluate this you need to know $g'(6)$, which you are not given.

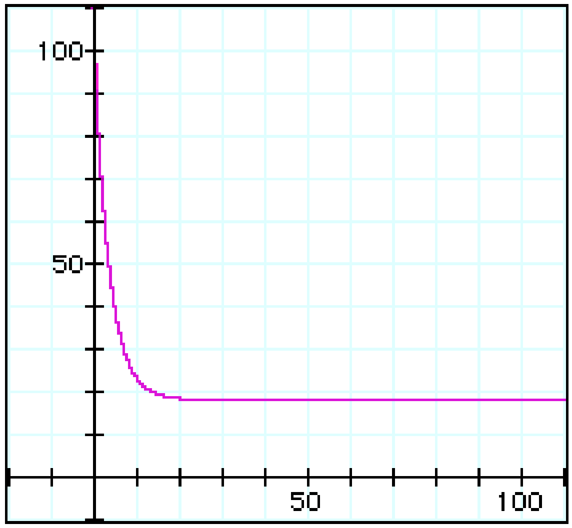
9. The only graph that could be a graph of the function is Graph B. The graph of the second derivative shows that the second derivative is negative when $x < 1$, positive when $1 < x < 3$, and negative when $x > 3$. This means that the original function would have to be:

x-values	Second derivative	Original function
x < 1	Negative	Concave down
1 < x < 3	Positive	Concave up
x > 3	Negative	Concave down

The only one of the four graphs that shows this pattern of concavity is Graph B.

10.(a) As $t \to \infty$, you would expect the egg to cool down to the same temperature as the water. This is 18°C.

10.(b) A graph showing temperature versus time is given below. Note the graph begins at a height of 98 (the initial temperature of the egg) and that as time goes by, the graph approaches a height of 18 (the temperature of the water).



10.(c) I would expect k to be a negative number. This is because the graph of temperature versus time decreases as time increases.

10.(d) $\lim_{t \to \infty} T(t) = C$.

10.(e) On one hand, you know that C is the limit of $T(t)$ as $t \to \infty$. On the other hand, you know that as $t \to \infty$, $T(t)$ approaches 18°C. Therefore, $C = 18$.

10.(f) Plugging $t = 0$ into the formula for $T(t)$ and setting the result equal to 98 gives: $18 + A = 98$. Therefore, $A = 80$.

10.(g) Plugging $t = 5$ and 38 for the temperature into the equation for $T(t)$ gives: $38 = 18 + 80 * e^{5k}$. Solving this equation for k gives: $k = -0.28$.

10.(h) In this problem you are trying to solve the equation: $20 = 18 + 80 * e^{-0.28 * t}$ to find t. If you do this then you get $t = 13$. The derivative of $T(t)$ is:

$$T'(t) = -0.28 * 80 * e^{-0.28t}.$$

Plugging in $t = 13$ gives: $T'(13) = -0.5881$ degrees centigrade per minute.