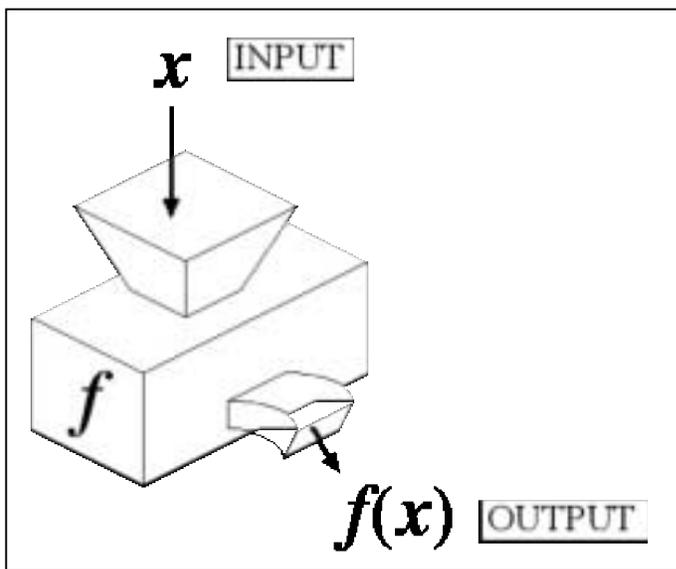

Review Notes – Functions

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.
2. The test will include twelve problems (each with multiple parts).
3. You will have 3 hours to complete the test.
4. You may use your calculator and one page (8" by 11.5") of notes on the test.
5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.
6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.
7. Good sources of help:
 - Section leaders' office hours (posted on Xa web site).
 - Math Question Center (during the reading period).
 - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.

1. Function Notation

You can make an analogy between a function and a machine (like a meat grinder). The purpose of this analogy is to link together the abstract symbols used in function notation with a mechanical device that you are already very familiar with. If you every get stuck on or confused by some function notation, try to think of what each symbol present would represent in “meat grinder terms.”

- x - this is the unprocessed meat that goes into the meat grinder.
- f - this is the name of the machine that is being used (the meat grinder itself)
- $f(x)$ - this is the stuff (ground meat) that comes out of the machine.

1.1 Representing changes to the inputs and outputs

Changes to what goes into the machine should be reflected in what comes out of the machine. For example, if you put less meat, $x - 1$, in there, then the output should be changed as well. The symbols that reflect this changed level of output are: $f(x - 1)$.

Because the change was made to the input into the function, the output reflects this change, but the alteration is a change in the symbols that appear *inside* the parentheses.

If, instead, you made a change to the output of the machine without making a change to the input, then this would be reflected by a change in the symbols used outside the brackets in function notation. For example, if the person grinding the meat started adding some “filler” like flour or bread crumbs to the meat as it came out of the grinder then this would be reflected in a change like: $f(x) + 1$.

This input is still the same, but the amount that comes out of the machine is now larger.

2. Transformations

In Math Xa we studied three ways to create new functions from old. The first was *arithmetic* – that is, adding, subtracting, multiplying or dividing two functions. The second was composition in which the output from one function became the input to a second function. The third way to create new functions from old is to perform some kind of transformation on the graph of an existing function.

When modifying an existing function with a transformation, the key points to bear in mind is these:

- When you make a change **inside** function notation, you are modifying the inputs or x -values. Therefore, when you make a transformation **inside** the function notation, you will affect the x -coordinates of points on the graph, and cause the graph to move, stretch or reflect in a **horizontal** fashion.
- When you make a change **outside** the function notation, you are modifying the outputs or y -values. Therefore, when you make a transformation **outside** the function notation, you will affect the y -coordinates of points on the graph, and cause the graph to move, stretch or reflect in a **vertical** fashion.

Modification of equation or function notation	Transformation that was performed
Multiply “inside” by -1	Reflection across y-axis
Multiply “outside” by -1	Reflection across x-axis
Add/subtract on “inside”	Horizontal translation
Add/subtract on “outside”	Vertical translation
Multiply/divide on “inside”	Horizontal stretch
Multiply/divide on “outside”	Vertical stretch

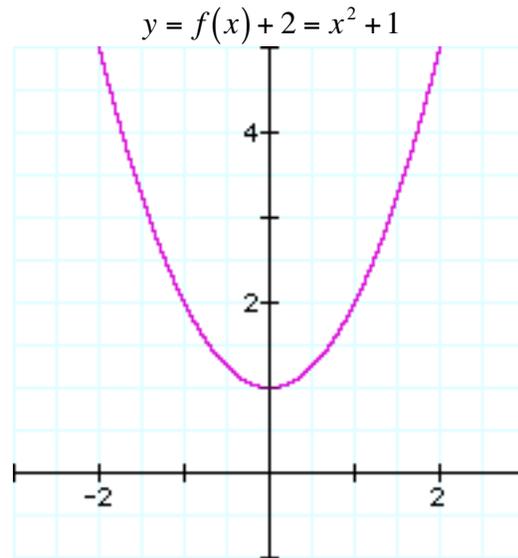
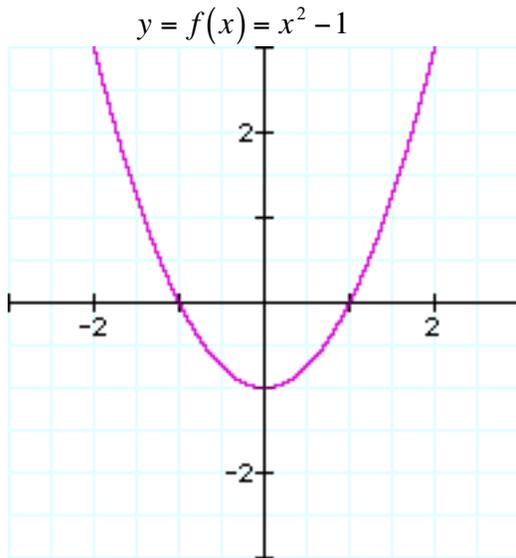
2.1 Vertical Translations

These are caused by:

- **adding** (to move the graph up) or
- **subtracting** (to move the graph down)

a number from the **outside** of the function.

2.1.1 Example of a Vertical Translation



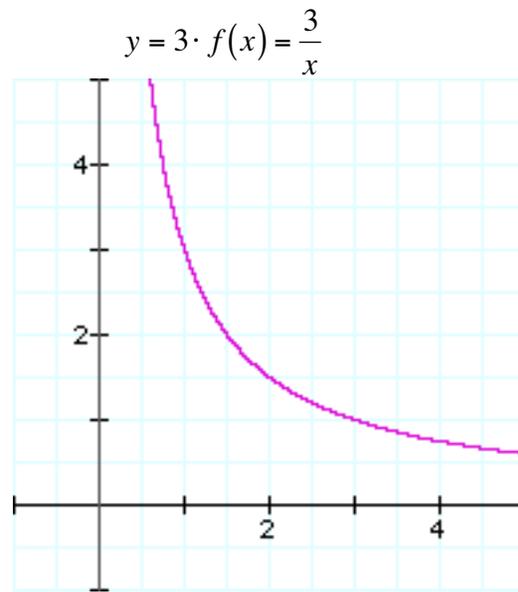
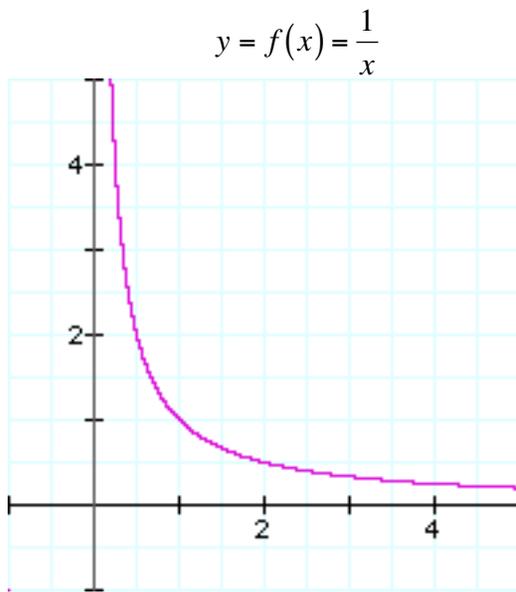
2.2 Vertical Stretches

These are caused by:

- **multiplying** by a number greater than one (to elongate the graph vertically) or
- **multiplying** by a number between zero and one (to compress the graph vertically)

on the **outside** of the function.

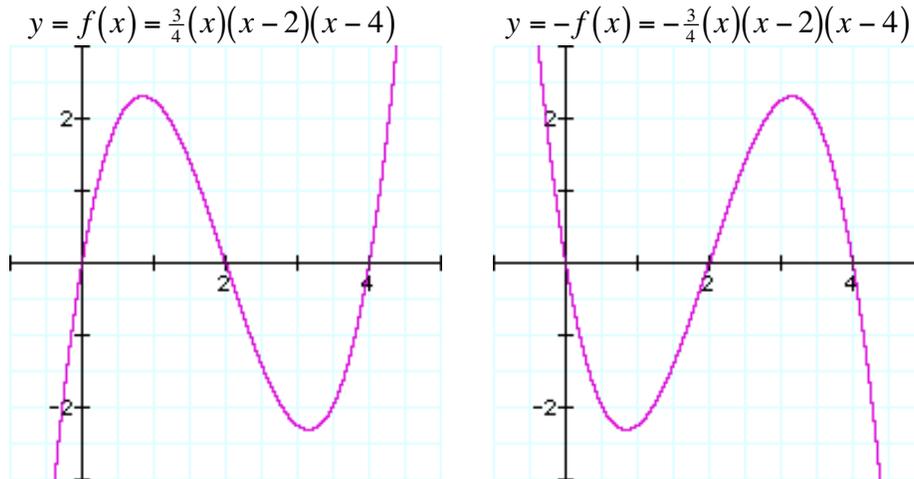
2.2.1 Example of a Vertical Stretch



2.3 Reflections across the x-axis

These are caused by **multiplying** the **outside** of the function by -1 .

2.3.1 Example of a Reflection across the x-axis



2.4 Horizontal Translations

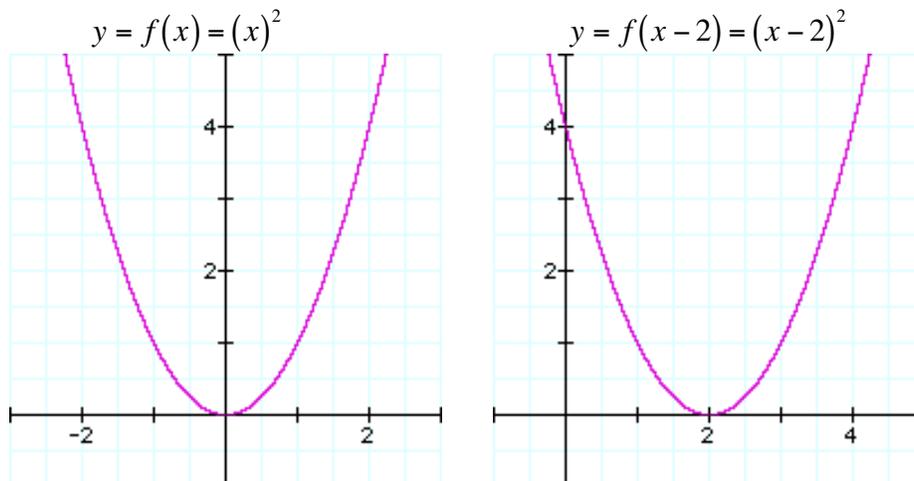
These are caused by:

- **adding** (to move the graph to the left) or
- **subtracting** (to move the graph to the right)

a number from the **inside** of the function.

NOTE: This is the reverse (add to move to the left, subtract to move to the right) of what most people intuitively expect.

2.4.1 Example of a Horizontal Translation



2.5 Horizontal stretches

These are caused by:

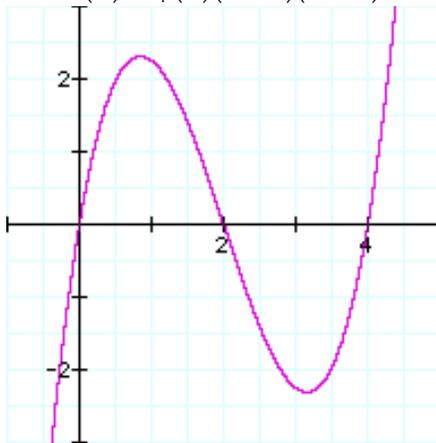
- **multiplying** by a number greater than one (to compress the graph horizontally) or
- **multiplying** by a number between zero and one (to elongate the graph horizontally)

on the **inside** of the function.

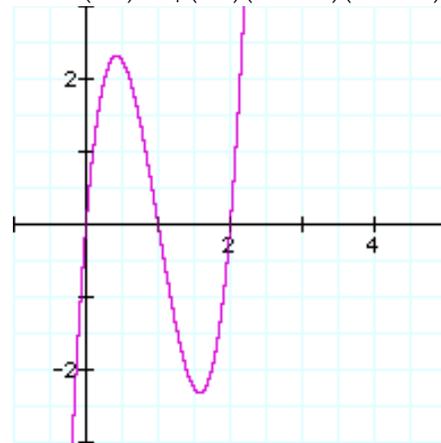
NOTE: This is the reverse (multiply by a number greater than one to compress, by a number smaller than one to elongate) of what most people intuitively expect.

2.5.1 Example of a Horizontal Stretch

$$y = f(x) = \frac{3}{4}(x)(x-2)(x-4)$$



$$y = f(2x) = \frac{3}{4}(2x)(2x-2)(2x-4)$$

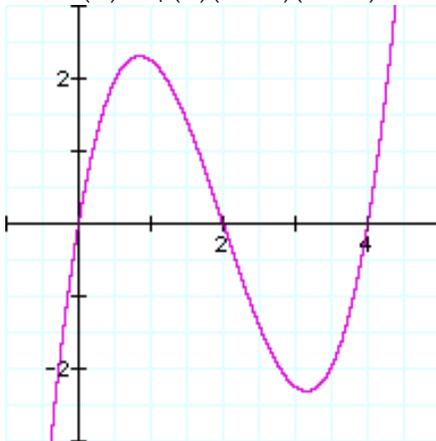


2.6 Reflections across the y-axis

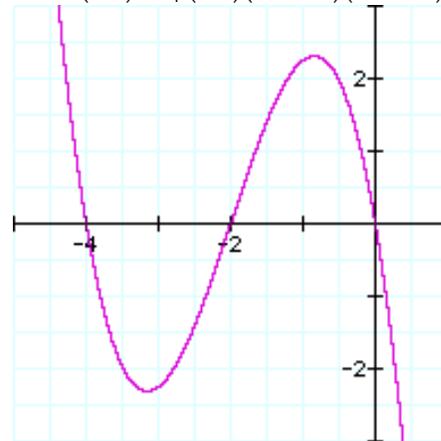
These are caused by **multiplying** the **inside** of the function by -1 .

2.6.1 Example of a Reflection across the y-axis

$$y = f(x) = \frac{3}{4}(x)(x-2)(x-4)$$



$$y = f(-x) = \frac{3}{4}(-x)(-x-2)(-x-4)$$

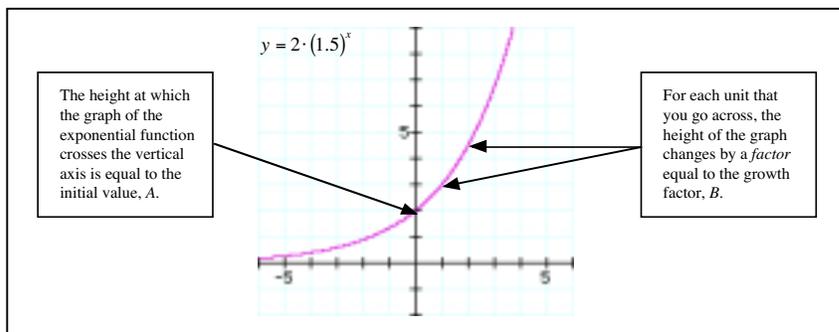


3. Exponential Functions

An **exponential function** is a function in which the independent variable appears in the exponent. For example, if x represents the independent variable and y represents the dependent variable then the equation of an exponential function usually resembles:

$$y = A \cdot B^x$$

where A is a number called the **initial value** and B is a number called the **growth factor**. An exponential function is completely determined by the values of A , the initial value, and B , the growth factor.



When an exponential function is graphed, the values of A and B influence the appearance of the graph. Consider the graph of the exponential function $y = 2 \cdot (1.5)^x$ which has initial value $A = 2$ and growth factor $B = 1.5$.

The four main ways that an exponential function's graph can appear are shown in the following table. When you are looking at the plot of a set of data, comparing the patterns you see in the data to the appearances of the graphs in the following table can help you to decide when an exponential function might do a good job of representing the trend in the data.

	$0 < B < 1$	$B > 1$
$A < 0$	<ul style="list-style-type: none"> Increasing Concave down $y = -1 \cdot (0.75)^x$	<ul style="list-style-type: none"> Decreasing Concave down $y = -1 \cdot (1.5)^x$
$A > 0$	<ul style="list-style-type: none"> Decreasing Concave up $y = 1 \cdot (0.75)^x$	<ul style="list-style-type: none"> Increasing Concave up $y = 1 \cdot (1.5)^x$

3.1 Example: Steps in Calculating a Formula for an Exponential Function

Find the equation of the exponential function,

$$y = A \cdot B^x$$

whose graph passes through the two points (1, 4) and (4, 8) you:

Solution

Step 1: Substitute the coordinates of the two points into the equation for an exponential function

$$8 = A \cdot B^4$$

$$4 = A \cdot B^1$$

Step 2: Find the quotient of the two equations, simplify and solve for B

$$\frac{8}{4} = \frac{A \cdot B^4}{A \cdot B^1} = \frac{B^4}{B^1} = B^{4-1} = B^3$$

$$B = (B^3)^{\frac{1}{3}} = (2)^{\frac{1}{3}} = 1.25992105$$

Step 3: Substitute one of the points and the value for B into the exponential equation and solve for A

$$4 = A \cdot (1.25992105)^1$$

$$A = 3.174802104$$

The completed equation is: $y = (3.174802104) \cdot (1.25992105)^x$

3.2 Example: Calculating a Formula for an Exponential Function

Strontium-90 is a radioactive substance. Strontium-90 is sometimes released during nuclear accidents (such as the 1986 Chernobyl disaster) and is regarded as a very dangerous substance because it is chemically similar to calcium, and can become incorporated into bone tissue. The mass, M , of strontium-90 remaining in a person's bones after T years is described by an exponential formula:

$$M = M_0 \cdot B^T,$$

where M_0 is the mass of strontium-90 that was absorbed into the person's bones when they were exposed to radioactivity, and B is a number.

Bone samples from the person indicated that 12 years after exposure, the person had 0.75 grams of strontium-90 in their body, and 30 years after exposure, the person had 0.488 grams of strontium-90 in their body. Find an equation for amount of strontium-90 as a Function of time.

Solution:

We are told that when $T = 12$, $M = 0.75$. Putting this together with the equation gives:

$$0.75 = M_0 \cdot B^{12} \quad \dots (1)$$

Likewise, we are told that when $T = 30$, $M = 0.488$, so that:

$$0.488 = M_0 \cdot B^{30} \quad \dots (2)$$

There are two equations (Equations (1) and (2)), and two unknowns, M_0 and B . The strategy is to combine the two equations in order to eliminate one of the variables. This can be accomplished here by division to cancel out the M_0 's:

$$\frac{0.488}{0.75} = \frac{M_0 \cdot B^{30}}{M_0 \cdot B^{12}} = \frac{B^{30}}{B^{12}} = B^{30-12} = B^{18}.$$

(The Laws for Exponents were used to combine the powers of B .) Noting that $0.488/0.75 = 0.65$, we are left with an equation that includes only one of the unknown quantities, B :

$$B^{18} = 0.65$$

$$B = (0.65)^{1/18} = 0.9764.$$

To find the numerical value of M_0 , we can plug $B = 0.9764$ into Equation (1) and solve for M_0 :

$$0.75 = M_0 \cdot (0.9764)^{12}$$

$$M_0 = 0.75 / (0.9764)^{12} = 0.9989.$$

So, the equation for the mass of strontium-90 as a function of time is:

$$M = 0.9989 \cdot (0.9764)^T.$$

4. Power Functions

A power function is a function in which the independent variable, x , is raised to a power. The equation of this kind of a function always resembles:

$$y = k \cdot x^p$$

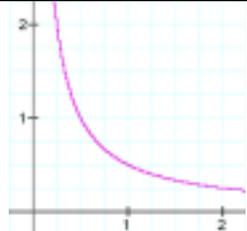
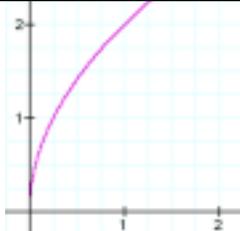
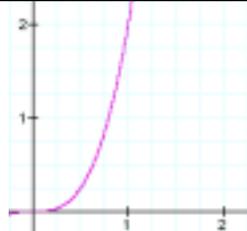
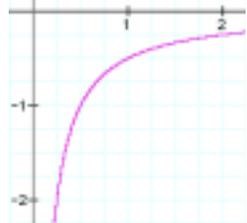
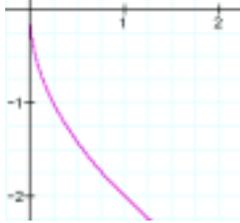
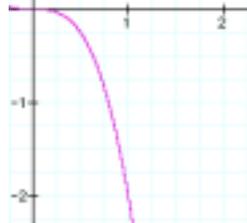
where x is the independent variable and y is the dependent variable. The symbols k and p represent numbers. The number k is called the **constant of proportionality**, and the constant p is called the **power**.

Although the equation for a power function looks a little like the equation for an exponential function, power functions and exponential functions are actually very different. In an exponential function, the independent variable x appears in the exponent of the equation. In a power function, the independent variable x is **not in the exponent** – instead, the exponent is a fixed number, p .

4.1 Graphs of Power Functions

The values of the power, p , and the constant of proportionality, k , determine the appearance of the graph of the power function:

$$y = k \cdot x^p.$$

	$p < 0$	$0 < p < 1$	$p > 1$
$k > 0$			
$k < 0$			

4.2 Vertical and Horizontal Asymptotes

When the power p in a power function is negative, you will have noticed that when x is close to zero, the height of the graph suddenly shoots up. Any place where the graph of a function suddenly shoots up without ever coming back down is called a **vertical asymptote**.

You might also have noticed the when the power p in a power function is negative, the graph gets closer and closer to the x -axis the further out from $x = 0$ that you go. Any place where the graph of a function levels out and gets closer and closer to a fixed height is called a **horizontal asymptote**.

To see why power functions sometimes have vertical and horizontal asymptotes, let's consider the example:

$$y = 3 \cdot x^{-1} = \frac{3}{x}.$$

When x is very close to zero, the equation determining y looks like:

$$y = \frac{3}{\text{a very small number}} = \text{a very large number overall.}$$

So, since the y -value gets very, very big, the height of the graph must shoot up.

On the other hand, when x is very close to zero, the equation determining y looks like:

$$y = \frac{3}{\text{a very large number}} = \text{a very small number overall.}$$

So, since the y -value gets very, very small, the height of the graph must drop closer and closer to the x -axis.

4.3 Example: Finding a Formula for a Power Function

- It is probably always best to use the PwrReg capability of a graphing calculator when trying to find the equation for a power function, although it is possible to determine an equation by hand so long as the points that you are given are “nice” enough.

One of the great triumphs of Sir Isaac Newton’s scientific career was his successful prediction of the return of Halley’s comet in 1682¹. Newton was so highly respected for his contributions to mathematics and science that he was laid to rest in Westminster Abbey (one of England’s holiest and most revered sites, generally reserved for England’s royal family).

In order to predict the return of Comet Halley, Newton had to relate the time that it took for an object to travel once around the sun to astronomical observations, as the only accurately known property of Halley’s comet (at least in the seventeenth century) was its average distance from the Sun.

Table 1 (below) gives the average distance from the Sun and orbital period (the amount of time needed to go once around the Sun) for six of the planets in the solar system.

Planet	Image	Mean distance from Sun (millions of miles)	Orbital period (days)
Mercury		36	88
Venus		67	225
Earth		93	365
Mars		142	687
Jupiter		484	4333
Saturn		866	10759

Table 1: Planetary Data.

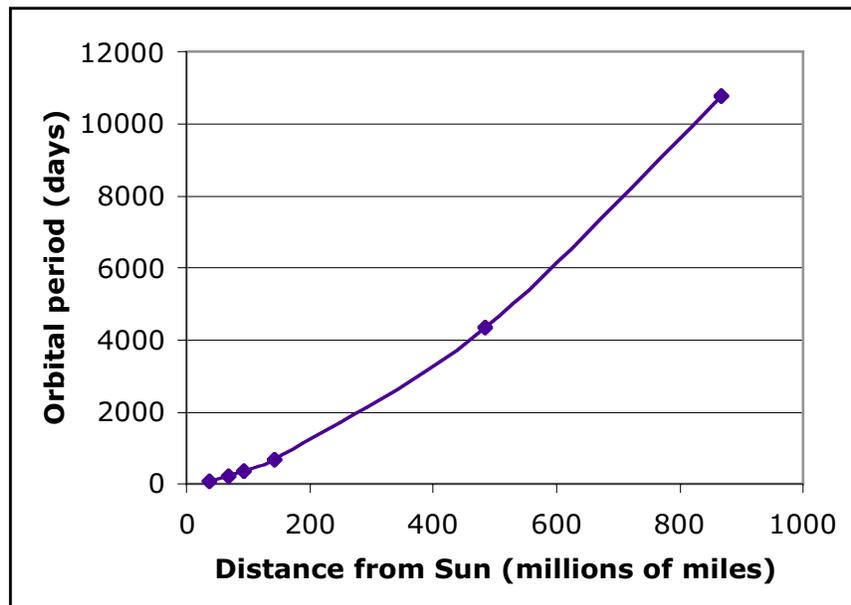
- (a) Given that Newton was trying to predict the orbital period of Halley’s comet, and that he knew the average distance from the Sun, which quantity would make a good choice for the independent variable, and which quantity would make a good choice for the dependent variable?

¹ Source: Halliday, D., Resnick, R. and Walker, J. (1997) “Fundamentals of Physics. Fifth Edition.” New York: John Wiley and Sons.

- (b) Use the STATPLOT capability of your calculator to produce a graph of the data given in Table 1. What sort of function would probably do a reasonable job of representing the trend in this data?
- (c) Use the regression capabilities of your calculator to find an equation to represent the relationship between orbital period and distance from the Sun.
- (d) The planet Pluto was discovered in 1930 by astronomer Clyde Tombaugh. Pluto's average distance from the Sun is 3660 miles. Predict the orbital period.

Solution

- (a) The average distance from the Sun would make the most sense for an independent variable, as this is the quantity whose value Newton actually knew. The orbital period would make the most sense for the dependent variable as this is the value that Newton did not know and was interested in predicting.
- (b) Plotting the points gives a graph that resembles the one shown below.



Given that the data points do not all lie on a straight line, and that the overall appearance of the plot is quite reminiscent of the graph of an exponential function graph. It could also be a power function (with power $p > 1$) as the graph looks as though it might go through the point $(0, 0)$.

- (c) If you perform exponential regression on the data points in Table 1, you obtain:

$$y = (179.9553842) \cdot (1.005242919)^x$$

where y is the orbital period, in units of days, and x is the mean distance from the Sun in units of millions of miles. If, instead, you perform power regression you get:

$$y = (1.4087) \cdot x^{1.4995}$$

The specific calculator commands (for a TI-83 calculator) that were used to obtain this formula are shown below.

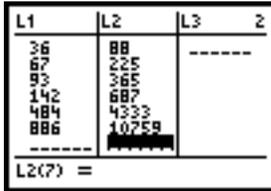


Figure 1a: Enter the data from Table 1 into the lists on your calculator.



Figure 1b: Press the [STAT] button and select the CALC menu. Use the arrow keys to select the PwrReg option.



Figure 1c: Press [ENTER]. The symbol L1 is obtained by pressing [2nd] [1], etc.

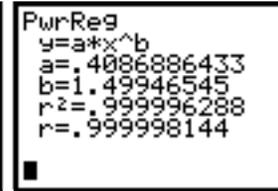


Figure 1d: Press [ENTER]. Your calculator should display the equation for a power function.

- (d) Plugging $x = 3660$ into the exponential equation gives: $y \approx 3.7 \times 10^{10}$ days.
 Plugging $x = 3660$ into the power equation gives: $y = 92,946.2$ days.

The prediction from the power function is much closer to the measured value of 90591 days.

5. Polynomial Functions

A polynomial function is any function of the form:

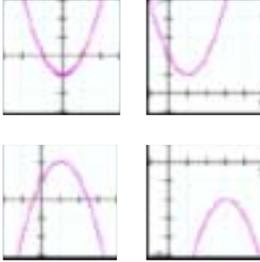
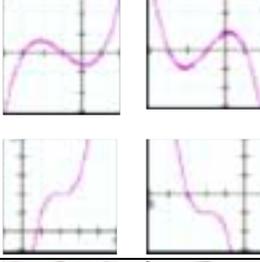
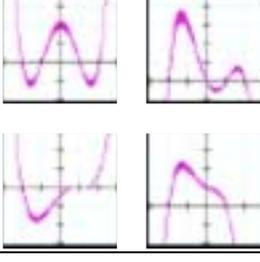
$$y = c_0 + c_1 \cdot x + c_2 \cdot x^2 + \dots + c_n \cdot x^n$$

where the powers of x must be integers. The largest power of x in the polynomial is called the **degree** of the polynomial.

5.1 Graphs of Polynomial Functions

The graphs of the different types of polynomial functions usually have quite distinct and recognizable shapes. Typical examples of the shapes of the polynomial functions (up to degree four) are shown in the table below.

Name	Typical appearance of graph	Equation (a, b, c, d, e are all constants)	Characteristics
Constant function		$y = a$	Flat, horizontal graph
Linear function		$y = ax + b$	Graph is a straight line

Quadratic function		$y = ax^2 + bx + c$	One “hump”
Cubic function		$y = ax^3 + bx^2 + cx + d$	May have two “humps” or an “inflection point”
Quartic function		$y = ax^4 + bx^3 + cx^2 + dx + e$	May have one or three “humps” or one “hump” and an “inflection point”

5.2 Finding Formulas for Polynomial Functions

Most graphing calculators can only fit a very limited number of polynomial functions to data. (For example, the TI-83 can only fit polynomials up to degree 4). If you are given a table of data and asked to find the polynomial function that “best fits” the data, and the “shape” that you see when you plot the data is reminiscent of a quadratic, cubic or quartic graph then polynomial regression is the way to go.

For polynomials with degrees greater than four, you can often use the graph to find an equation for the polynomial function.

The basic procedure is to:

1. Locate the “x-intercepts” or zeros of the polynomial function.
2. Determine the multiplicity of each zero.
3. Write down the “factored form” of the polynomial.
4. Use a point on the graph of the polynomial to determine the constant of proportionality, k .

5.2.1 Example: Finding a Formula for a Polynomial from a Graph

Find a formula for the polynomial function shown in Figure 2.

Solution

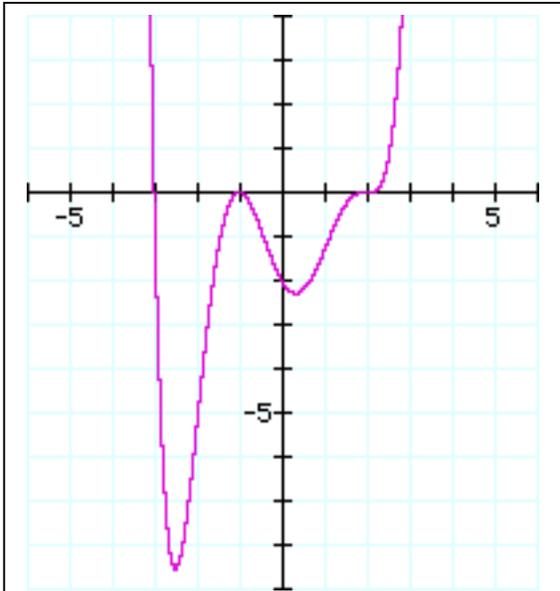


Figure 2: Example for finding the equation of a polynomial function by hand.

Step 1: Locate the zeros

Inspection of Figure 2 shows that the zeros of this polynomial are located at $x = -3$, $x = -1$ and $x = 2$.

Step 2: Determine the Multiplicity of Each Zero

The multiplicity of the zero is determined by the appearance of graph near the zero (See Figure 3).

- If the graph looks as though it just cuts cleanly through the x -axis, then the zero has multiplicity one (see Figure 3(a)).
- If the graph looks like a quadratic and just touches the x -axis without cutting through, then the zero has multiplicity two (see Figure 3(b)).
- If the graph looks like a cubic and has an inflection point as it cuts through the x -axis, then the zero has multiplicity three (see Figure 3(c)).

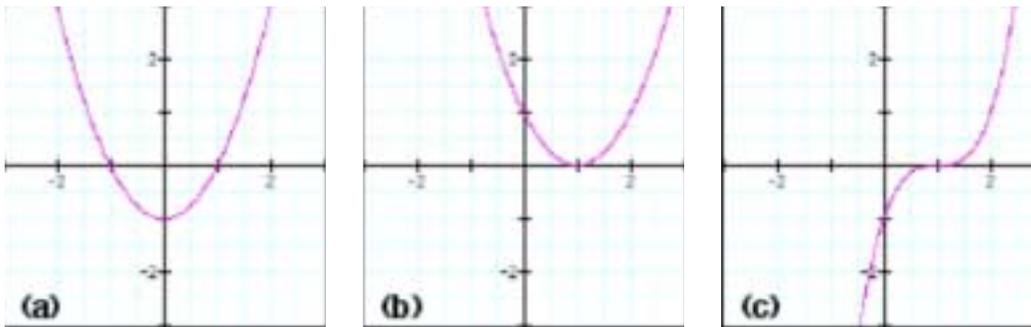


Figure 3: Multiplicity of zeros. (a) Two zeros, each of multiplicity one. (b) One zero of multiplicity two. (c) One zero of multiplicity three.

For the function graphed in Figure 2, the multiplicity of the zeros are:

Zero located at ...	Multiplicity
$x = -3$	1
$x = -1$	2
$x = 2$	3

Step 3: Determine the Factored Form of the Polynomial

The factored form of the polynomial is an equation of the form:

$$y = k \cdot (x - z_1)^{m_1} \cdot (x - z_2)^{m_2} \dots (x - z_n)^{m_n},$$

where k is called the “constant of proportionality, z_1, z_2, \dots, z_n are the zeros of the polynomial function, and m_1, m_2, \dots, m_n are the multiplicities of the zeros. The factored form of the polynomial from Figure 2 is:

$$y = k \cdot (x + 3) \cdot (x + 1)^2 \cdot (x - 2)^3.$$

Step 4: Determine the Constant of Proportionality, k

The idea here is to locate the x and y coordinate of a point that is on the graph of the polynomial function, but which is not one of the zeros of the polynomial function. The x and y are substituted into the factored form, allowing k to be found.

From Figure 2, the point $(0, -2)$ lies on the graph of the function. Substituting this into the factored form gives: $k = 1/12$. Therefore, the equation for the polynomial function whose graph is show in Figure 2 is:

$$y = \frac{1}{12} \cdot (x + 3) \cdot (x + 1)^2 \cdot (x - 2)^3.$$

5.3 Two Final Words of Warning

Polynomial functions can do a really good job of producing a graph that matches a limited number of points, and when you do polynomial regressions on a calculator, you often get very high correlation coefficients (especially for cubic and quartic regression).

As a general guideline, it is best to use the **simplest** polynomial function that you possibly can to represent the trends in a data set. For example, if your data set shows one big hump and a few very minor wiggles then the main pattern that is there could be adequately described using a quadratic function. The presence of the few minor wiggles, however, will mean that a quartic regression will give a higher value of the correlation coefficient than a quadratic regression will. However, a quartic function is much more complicated and ungainly than a quadratic function so what you gain in having a slightly more accurate match to the data will be offset by the fact that a quartic function is much more ungainly and difficult to manipulate.

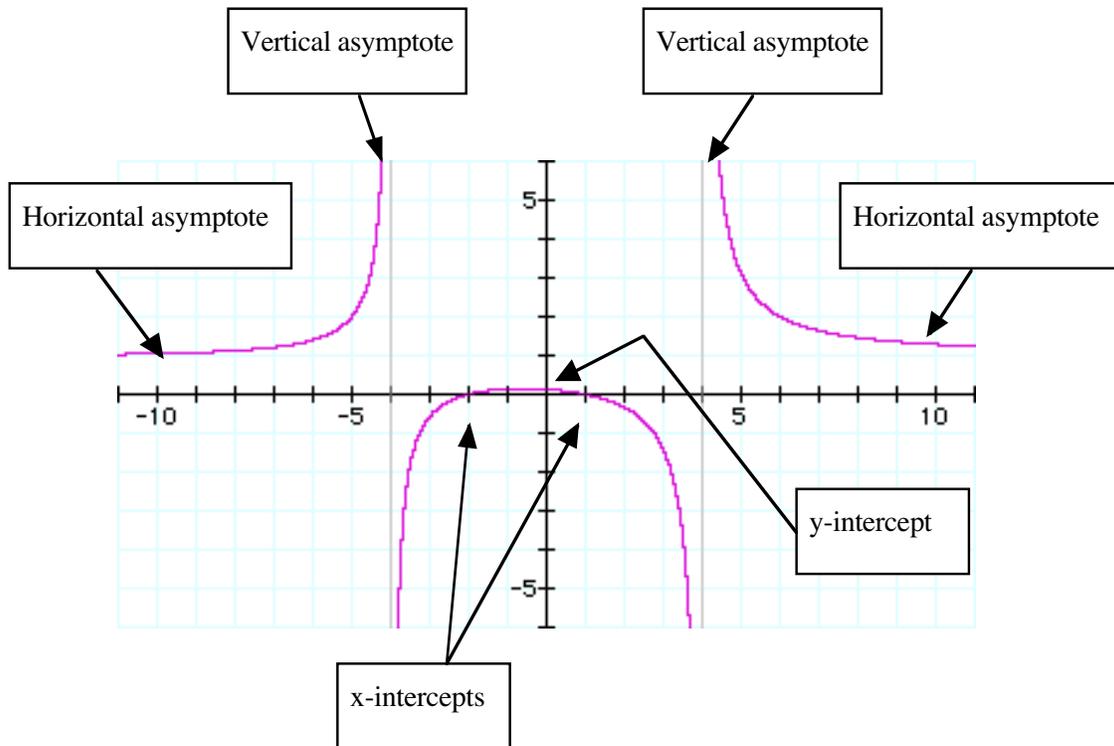
The second point is that while polynomial functions can do a good job of matching the pattern shown by a limited number of data points, they usually do not do a very good job of representing the situation at all times. For example, in the biomedical data lab you used a quadratic function to describe the heart rate of a seal. Near the data points that you had, the values of heart rate given by the quadratic function were very sensible. However, when you tried to plug a very big x -value into the quadratic formula you got a negative value for the hear rate, which isn't very realistic.

6. Rational Functions

A rational function is a quotient of two polynomials. For example,

$$f(x) = \frac{x^2 + x - 2}{x^2 - 16} = \frac{(x - 1) \cdot (x + 2)}{(x + 4) \cdot (x - 4)}.$$

The main features of the graph of a rational function that you have to pay attention to are: vertical asymptotes, horizontal asymptotes, x -intercepts and y -intercepts.



6.1 Relationship Between the Algebraic Structure of a Rational Function's Formula and the Appearance of Its Graph

- When the polynomial in the **denominator** of the rational function has a zero, the graph of a rational function may have a **vertical asymptote**.
- If there is a “compensating factor” in the numerator of the rational function, then there will be no vertical asymptote.
- If there is no “compensating factor” in the numerator, then the graph of the rational function will have a vertical asymptote.
 - By carefully examining the equation of the rational function, you can calculate the right and left hand limits of the function as it approaches the vertical asymptote and decide whether each of the “arms” of the graph rises up to $+\infty$ or dives down to $-\infty$.
- When the polynomial in the **numerator** of the rational function has a zero, this causes the graph to have an **x -intercept**.
- The **y -intercept** of the graph of the rational function is the y -value that you get when you plug $x = 0$ into the equation for the rational function.

6.1.1 Example: Determining the Appearance of the Graph of a Rational Function Near the Points Where the Function is Not Defined

In this example, the function $f(x)$ will always refer to the rational function:

$$f(x) = \frac{x^3 + 2x}{(x+2)(x-1)}.$$

- (a) Locate any points at which $f(x)$ is not defined.
- (b) Determine the behavior of the function $f(x)$ near any points where it is not defined. (That is calculate the limit of $f(x)$ as x gets close to the point.)

Solution

- (a) From the algebraic structure of the equation, these “hot spots” are likely going to be around $x = -2$ and $x = 1$ as these are the two x -values that will make the denominator equal to zero.
- (b) We need to determine what happens near these two points. We will consider each point in turn.

I. Behavior of $f(x)$ near $x = -2$.

If we just try to plug $x = -2$ into the formula for $f(x)$ we'll get $0/0$, which is not very meaningful. Therefore, it will probably be to our advantage to see whether we can factor or manipulate the formula for $f(x)$ so that it will reveal some more of its secrets.

$$f(x) = \frac{x^3 + 2x}{(x+2)(x-1)} = \frac{(x+2) \cdot x^2}{(x+2) \cdot (x-1)}.$$

So, there is a factor of $(x+2)$ in the numerator that will balance the effects of the factor of $(x+2)$ in the denominator of the function. The presence of this “compensating factor” in the numerator of the function means that the value of $f(x)$ will not approach $\pm\infty$ as $x \rightarrow -2$. Instead, as $x \rightarrow -2$,

$$f(x) = \frac{(x+2) \cdot x^2}{(x+2) \cdot (x-1)} \approx \frac{(-2)^2}{-2-1} = \frac{-4}{3}$$

so that the limit of $f(x)$ as $x \rightarrow -2$ is equal to $-4/3$.

II. Behavior of $f(x)$ near $x = 1$.

If you carefully examine the algebraic structure of the formula for $f(x)$,

$$f(x) = \frac{x^3 + 2x}{(x+2)(x-1)} = \frac{(x+2) \cdot x^2}{(x+2) \cdot (x-1)}$$

you can see that there is no factor in the numerator of the function to compensate for the factor of $(x-1)$ in the denominator. Therefore, it will probably be the case that the value of $f(x)$ will become infinite as x approaches 1.

(This feature of a graph – a place where the graph either shoots up to $+\infty$ or dives down to $-\infty$ at a particular, finite x -value is called a **vertical asymptote**.)

Will the value of $f(x)$ approach $+\infty$ or $-\infty$ as x approaches 1? Often, $f(x)$ does different things on each side of a vertical asymptote, so to be safe you need to do the left hand limit ($x \rightarrow 1^-$) and the right hand limit ($x \rightarrow 1^+$) separately.

Left hand limit: When x is slightly less than 1, the value of $f(x)$ will be:

$$f(x) = \frac{(x+2) \cdot x^2}{(x+2) \cdot (x-1)} = \frac{(\text{approx. } 3) \cdot (\text{approx. } 1)}{(\text{approx. } 3) \cdot (\text{negative number very close to zero})} = \text{large negative}$$

So, as x approaches 1 from the left, the limit of $f(x)$ will be $-\infty$.

Right hand limit: When x is slightly greater than 1, the value of $f(x)$ will be:

$$f(x) = \frac{(x+2) \cdot x^2}{(x+2) \cdot (x-1)} = \frac{(\text{approx. } 3) \cdot (\text{approx. } 1)}{(\text{approx. } 3) \cdot (\text{positive number very close to zero})} = \text{large positive}$$

So, as x approaches 1 from the right, the limit of $f(x)$ will be $+\infty$.

6.2 Horizontal Asymptotes

- The existence/non-existence of **horizontal asymptotes** can be decided by comparing the highest power of x in the numerator of the rational function to the highest power of x in the denominator of the rational function.

The reason that you can just focus on the highest power of x in the numerator and denominator of the rational function stems from the fact that horizontal asymptotes are features of the graph that appear when you are a long way away from the origin. That is, they appear when the x -values are very large (either very large positives or very large negatives).

When x is very large, the highest power of x will be – by far – the largest quantity in either the numerator or the denominator. So, when x is very large, the value that the rational function is approaching can be decided along the lines of reasoning shown below.

$$f(x) = \frac{3 \cdot x^2 + x - 2}{x^2 - 16} = \frac{3 \cdot x^2 + (\text{peanuts by comparison})}{x^2 - (\text{peanuts by comparison})} \approx \frac{3 \cdot x^2}{x^2} = 3.$$

So, the limit of this particular $f(x)$ as $x \rightarrow \infty$ would be equal to 3, and the graph of $y = f(x)$ will level out to resemble a horizontal line of height 3 when x is really, really big.

In general:

- If the highest power of x in the numerator *equals* the highest power of x in the denominator, then the graph will show a **non-zero horizontal asymptote**. The height of the asymptote will be the ratio of the highest power terms from numerator and denominator.
- If the highest power of x in the numerator is *less than* the highest power of x in the denominator, then the graph will show a **horizontal axis at height zero – i.e. the x -axis will be the horizontal asymptote**.

- If the highest power of x in the numerator is *greater* than the highest power of x in the denominator, then the graph will show **no horizontal asymptotes**.

6.3 Example: Determining the Horizontal and Vertical Asymptotes of a Rational Function

Locate the horizontal and vertical asymptotes of the rational function:

$$f(x) = \frac{7 \cdot x^2}{x^2 + x - 2} = \frac{7 \cdot x^2}{(x + 2)(x - 1)}$$

and use these to sketch a graph of the rational function.

Solution

First of all, there will be **vertical asymptotes** at $x = -2$ and $x = +1$. This is because there are no suitable factors in the numerator of this rational function to “compensate” for the factors of $(x + 2)$ and $(x - 1)$ in the denominator.

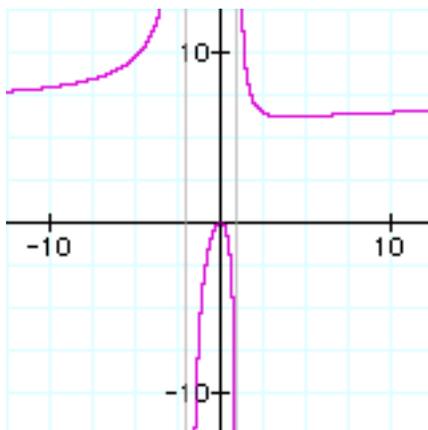
To determine what the “arms” of the graph do around these vertical asymptotes, you can calculate the right and left hand limits of $f(x)$ at $x = -2$ and $x = +1$. The results of these calculations are as follows:

	Left hand limit	Right hand limit
$x = -2$	$+\infty$	$-\infty$
$x = +1$	$-\infty$	$+\infty$

As the numerator of the rational function is $7 \cdot x^2$, the only point where the numerator of the rational function will be equal to zero is at the point $x = 0$.

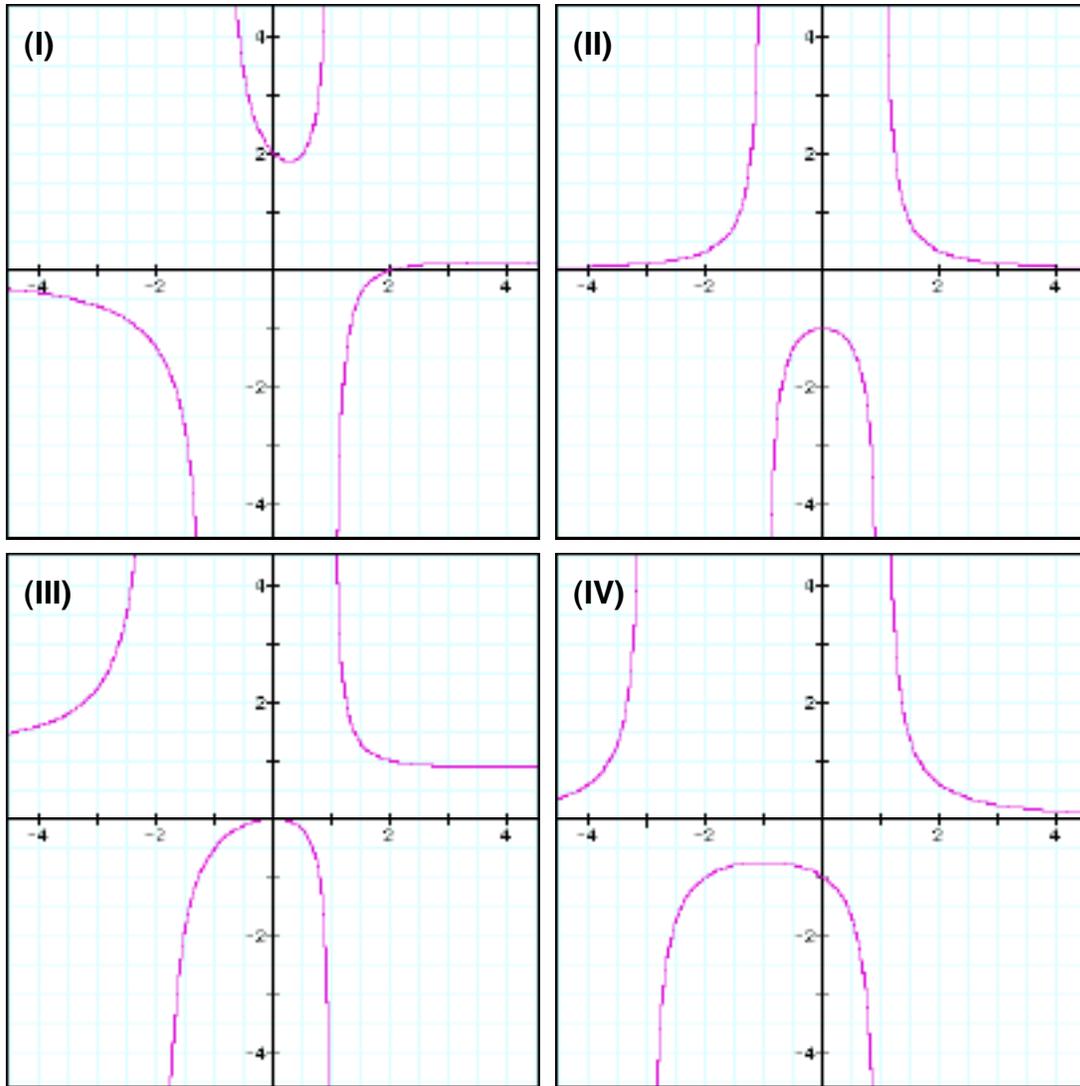
To determine whether there is a **horizontal asymptote** or not, notice that the highest power of x in the numerator is x^2 , and the highest power of x in the denominator is also x^2 . The ratio of these two x^2 terms is 7, so the graph will have a horizontal asymptote at height 7.

Putting all of this information together, the graph of this rational function $y = f(x)$ will resemble something like the following:



6.3 Example: Match the Graphs and Formulas for Rational Functions

Each of the graphs (I)-(IV) shows the graph of a rational function. Equations (A)-(D) give the equations for four rational functions. Match the graphs with the equations.



(A) $g(x) = \frac{x^2}{(x-1) \cdot (x+3)}$

(B) $h(x) = \frac{1}{(x-1) \cdot (x+1)}$

(C) $p(x) = \frac{x-2}{(x-1) \cdot (x+1)}$

(D) $q(x) = \frac{3}{(x-1) \cdot (x+3)}$

Solution:

(A) III (B) II (C) I (D) IV