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## Review Notes – Limits and Definition of Derivative

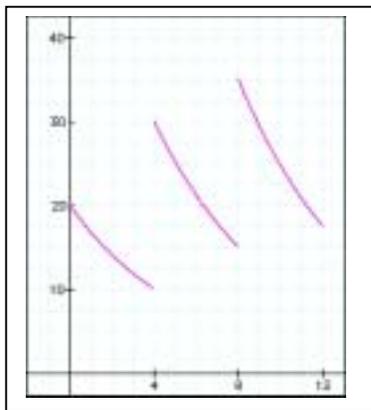
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Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.
  2. The test will include twelve problems (each with multiple parts).
  3. You will have 3 hours to complete the test.
  4. You may use your calculator and one page (8" by 11.5") of notes on the test.
  5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.
  6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.
  7. Good sources of help:
    - Section leaders' office hours (posted on Xa web site).
    - Math Question Center (during the reading period).
    - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.
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### 1. Left Hand, Right Hand and Overall Limits

#### 1.1 The Idea of a Limiting Value



The manufacturers of the popular morphine solution Roxanol suggest a dose of 20ml initially, followed by 20ml every four hours. The graph shows the amount of morphine sulfate that a patient receiving this treatment would have in their bloodstream as a function of time.

The lethal dose of Roxanol for the average person is about 180 ml. Is a patient who receives the suggested dose of Roxanol in any danger?

#### Solution

To try to answer this question, we can try to work out the amount of Roxanol present in the patient's body as time goes on. The numbers are shown in the following table.

Time (hours)	0	4	8	12	16	20	24
Morphine sulfate (mg)	20	30	35	37.5	38.75	39.375	39.6875

Some points to observe from this situation are:

- As time gets larger and larger, the amount of morphine sulfate gets closer and closer to 40.
- The amount of morphine sulfate never actually reaches 40, but if you keep going long enough, you can get the amount of morphine sulfate to be as close to 40 as you like.
- The value of 40 is the **limit or limiting value** of the amount of morphine sulfate in the patient's blood stream.

The limiting value is like the value of the dependent variable that you'd get if it were somehow possible to keep going and going for an infinite amount of time. (This is a limit or limiting value that occurs when the independent variable (in this case, time) keeps going and going towards infinity.)

If time is given by  $t$  and the amount of Roxanol by  $M(t)$  then the symbols that are normally used to describe this situation are:

$$\lim_{t \rightarrow \infty} M(t) = 40$$

## 1.2 Left Hand and Right Hand Limits

The morphine sulfate graph given in Section 1.1 was not (strictly speaking) complete as it did not specify exactly what was happening in the places where the graph suddenly jumped up. A more complete version of that graph is shown below<sup>1</sup>.



<sup>1</sup> I have assumed that the injection is always given exactly at the correct time – if you know about the physiology of drug absorption into the body you could probably successfully argue that the pattern of “open” and “filled in” endpoints should be reversed on the graph.

The **left hand limit** is the height on the graph that you *expect* to get to when you flow along the graph, heading for the point that you are interested in from the left side. For example, if you flowed along the morphine sulfate graph towards  $t = 4$  from the left side, you would expect to reach a height of 10 as you got closer and closer to  $t = 4$ . In mathematical symbols this **left hand limit** is written:

$$\lim_{t \rightarrow 4^-} M(t) = 10$$

The “-“ superscript that appears on the 4 shows that this is a left hand limit.

The **right hand limit** is the height on the graph that you *expect* to reach when you flow along the graph, approaching the point that you are interested in from the right side. For example, if you flowed along the morphine sulfate graph towards  $t = 4$  from the right side, you would expect to reach a height of 30 as you got closer and closer to  $t = 4$ . In mathematical symbols this **right hand limit** is written:

$$\lim_{t \rightarrow 4^+} M(t) = 30$$

The “+” superscript that appears on the 4 shows that this is a right hand limit.

### 1.3 Overall Limits

In situations where you are looking at the left and right hand limits of a function at a finite  $x$ -value, there are essentially two things that can happen:

- The left hand limit equals the right hand limit, or,
- The left hand limits does not equal the right hand limit.

In the case where the left hand limit and the right hand limit both exist, then normally one simply says that the *limit of the function exists* at the point in question, and that the value of the limit is the same as the common value of the left and right hand limits. The complete list of all situations that could possibly occur when looking at left hand and right hand limits (and trying to interpret them) is given in the following table.

Left hand Limit	Right hand Limit	Interpretation for Limit
L	$\pm\infty$	Does not exist
$\pm\infty$	L	Does not exist
L	M	Does not exist
L	L	Limit exists and is equal to L
$+\infty$	$+\infty$	Limit exists and is equal to $+\infty$
$-\infty$	$-\infty$	Limit exists and is equal to $-\infty$
$+\infty$	$-\infty$	Does not exist
$-\infty$	$+\infty$	Does not exist

Where L and M are finite and  $L \neq M$ .

## 2. Calculating Limits Using the Algebraic Structure of a Function

Generally speaking, when the denominator of a function approaches zero, the function has no value and the graph of the function has a vertical asymptote. Let's check the graph of the function

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$$

Near the point  $x = 1$ . If you graph this on your calculator, the following window settings will display the part of the graph that we are interested in.

- $x_{\min} = 0$
- $x_{\max} = 2$
- $y_{\min} = 1$
- $y_{\max} = 3$ .

If you enter the formula for  $f(x)$  correctly into your calculator, you should see something that looks like a parabola with a point missing in the center. This is quite a bit different from the vertical asymptote that you might have expected to see. The function  $f(x)$  is undefined at  $x = 1$  but there was no vertical asymptote on the graph of  $y = f(x)$ . (The reason the point is missing in the center is because  $x = 1$  is not in the domain of the function  $f$  so the calculator does not evaluate the function there.) What is it about the function

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$$

that allows it to be undefined at  $x = 1$  and allows it to avoid having a vertical asymptote at  $x = 1$ ?

Observe that:

$$x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1).$$

You might be tempted to go ahead and write something along the following lines:

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1} = \frac{(x - 1)(x^2 + 1)}{x - 1} = x^2 + 1,$$

and from this conclude (incorrectly) that  $f(1) = 2$ . This simplification is valid at all point **except**  $x = 1$ , which (unfortunately) is the point we are interested in. So, when  $x$  is close to – but not equal to – one, we have that  $f(x) = x^2 + 1$ . This means that when  $x$  is close to one,  $f(x)$  will be close to  $1 + 1^2 = 2$ .

The main thing that saves the graph of  $f(x)$  from having a vertical asymptote at  $x = 1$  is the fact that the denominator,  $x - 1$ , which approaches zero as  $x \rightarrow 1$ , is perfectly balanced by a “twin” factor of  $(x - 1)$  in the numerator which approaches zero in exactly the same way as the denominator does when  $x \rightarrow 1$ . If there were no perfect twin factor on the top line of the function to balance the  $x - 1$  in the denominator then the graph of  $f(x)$  would have had a vertical asymptote at  $x = 1$ .

### 2.1 Calculating Limits for Functions Defined by Equations

In addition to finding the values of limits by examining graphs and performing numerical calculations, you can often deduce the value of a limit by analyzing the symbolic representations of the function. We will concentrate on the two kinds of limits: limits as  $x \rightarrow \pm\infty$  and limits at finite values of  $x$ . The reasoning processes for analyzing the behavior of the function in these two kinds of limit situation are demonstrated through the following two examples:

### 2.1.1 Example: A Limit as $x \rightarrow \pm\infty$

Calculate the limit of  $f(x) = 1 - (0.5)^x$  as  $x \rightarrow \pm\infty$ .

#### Solution

**Limit of  $f(x) = 1 - (0.5)^x$  as  $x \rightarrow +\infty$**

The exponential function:  $y = (0.5)^x$  is a decreasing function. As  $x$  gets really, really big this exponential gets closer and closer to zero. So,  $f(x) = 1 - (0.5)^x$  gets closer and closer to  $1 - 0 = 1$ .

**Limit of  $f(x) = 1 - (0.5)^x$  as  $x \rightarrow -\infty$**

The exponential function:  $y = (0.5)^x$  is a decreasing function. Therefore, as  $x$  gets larger,  $(0.5)^x$  gets smaller. Conversely, when  $x$  gets more and more negative,  $(0.5)^x$  gets larger and larger. So, as  $x \rightarrow -\infty$ ,  $(0.5)^x$  will get very large and  $f(x) = 1 - (0.5)^x$  will resemble  $1 - (\text{very large number})$ , so the limit will be  $-\infty$ .

### 2.1.2 Example: Limits at Finite $x$ -values

Calculate the limit of  $g(x) = \frac{x+1}{x^2-1}$  as:

- (I)  $x \rightarrow 1^+$
- (II)  $x \rightarrow 1^-$
- (III)  $x \rightarrow -1$ .

#### Solution

**The limit of  $g(x) = \frac{x+1}{x^2-1}$  as  $x \rightarrow 1^+$**

When  $x$  is close to 1, the numerator of  $g(x)$  is basically equal to 2, and the denominator is very close to zero. If you are approaching from the right, then  $x > 1$ , so the denominator  $x^2 - 1 > 0$ . This means that when  $x$  is slightly larger than 1,

$$g(x) \approx \frac{2}{\text{very\_small\_positive\_number}} = \text{very\_large\_positive\_number}.$$

**The limit of  $g(x) = \frac{x+1}{x^2-1}$  as  $x \rightarrow 1^-$**

When  $x$  is close to 1, the numerator of  $g(x)$  is basically equal to 2, and the denominator is very close to zero. If you are approaching from the left, then  $x < 1$ , so the denominator  $x^2 - 1 < 0$ . This means that when  $x$  is slightly less than 1,

$$g(x) \approx \frac{2}{\text{very\_small\_negative\_number}} = \text{very\_large\_negative\_number}.$$

The limit of  $g(x) = \frac{x+1}{x^2-1}$  as  $x \rightarrow -1$

When  $x$  is close to  $-1$ , both numerator and denominator are very close to zero. This is a very confusing situation, and some care and finesse are required to make sense of it. Note that the polynomial in the denominator can be factored to give:

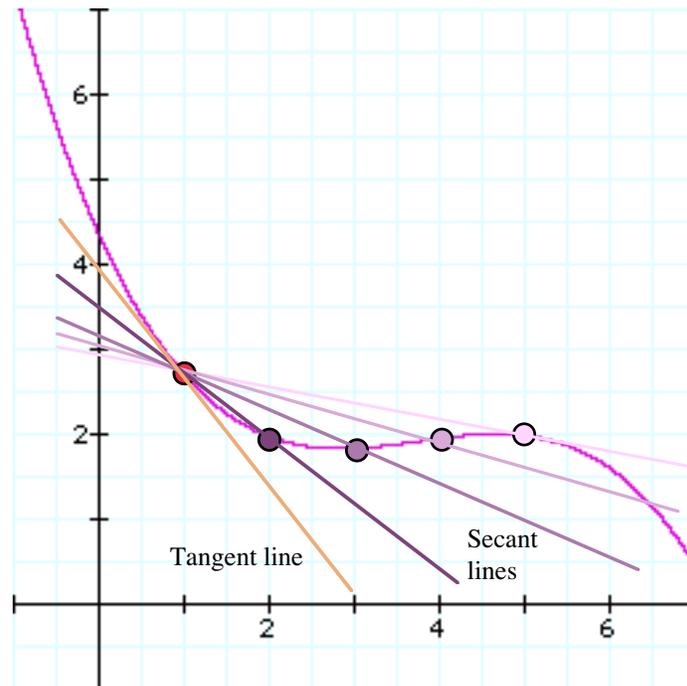
$$g(x) = \frac{x+1}{(x+1)(x-1)} = \frac{x+1}{x+1} \cdot \frac{1}{x-1} \approx \frac{x+1}{x+1} \cdot \frac{1}{-2}$$

when  $x$  is very close to  $-1$ . It is the presence of the factors of  $(x+1)$  in *both* the numerator *and* denominator that prevent  $g(x)$  from having a vertical asymptote at  $x=-1$ , as the very tiny value in the denominator is precisely balanced by an equal, tiny value in the numerator. When these two tiny values have balanced, all that remains is  $-1/2$ .

### 3. The Limit Definition of the Derivative

#### 3.1 The Tangent Line as the Limit of Secant Lines

The instantaneous rate of change of a function is the same as the slope of the tangent line to the graph of the function at whatever point you are interested in. If you find secant lines over smaller and smaller intervals, then when the intervals become very short, the secant line starts to resemble a tangent line.



The slope of the secant line joining the points  $(a, f(a))$  and  $(b, f(b))$  is given by

$$\text{Slope} = \frac{f(b) - f(a)}{b - a}$$

What we are doing as we bring the secant line closer and closer to the tangent line is keeping the point  $(a, f(a))$  fixed and bringing the other point  $(b, f(b))$  closer in. As you can see from the diagram, as the point  $(b, f(b))$  is brought closer and closer to  $(a, f(a))$  the slope of the secant line gets closer and closer to the slope of the tangent line.

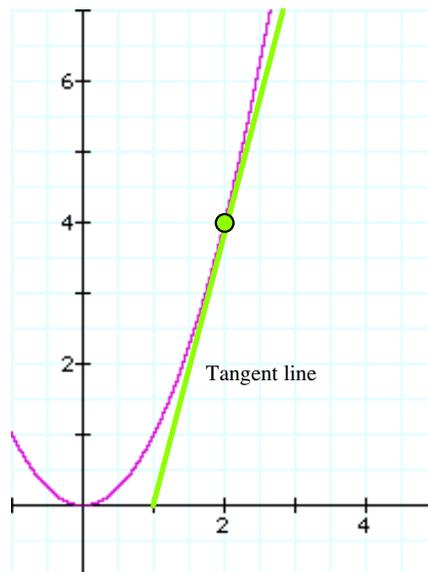
To obtain the slope of the tangent line, you can calculate the limit of:

$$\text{Slope} = \frac{f(b) - f(a)}{b - a}$$

as  $b \rightarrow a$ . The numerical value of this limit is called the **derivative** of the function  $f(x)$  at the point  $x = a$ .

### 3.2 A Numerical Example of Calculating the Derivative

The objective of this example will be to calculate the derivative of the function  $f(x) = x^2$  at the point  $x = 2$ . The number that we get will be the slope of the tangent line that just touches the graph of  $y = x^2$  at the point where  $x = 2$ .



#### Solution

First of all, let's try just doing some average rates of change over very small intervals near  $x = 2$  to see if we can guess the slope of the tangent line from the entries in the table.

Two points for secant line	Average rate of change of $f(x) = x^2$
$X = 2$ to $x = 2.1$	4.1
$X = 2$ to $x = 2.01$	4.01
$X = 2$ to $x = 2.001$	4.001
$X = 2$ to $x = 2.0001$	4.0001

The average rate of change seems to get closer and closer to a value of 4 as we make the interval shorter and shorter. Therefore, the limit of the average rate of change (as the distance between the two points that form the secant line shrinks down to zero) will be equal to 4. The slope of the tangent line to the function  $f(x) = x^2$  at the point  $x = 2$  is, therefore, 4.

You can also reach this conclusion by setting up a difference quotient for the slope of the secant line and “reading the algebra.” The slope of Tangent Line at  $x = 2$  is given by:

$$\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

Let’s write out what the difference quotient is, and try to simplify it algebraically as much as possible.

$$\text{Difference Quotient} = \frac{f(2+h) - f(2)}{h} = \frac{(2+h)^2 - 2^2}{h} = \frac{4 + 4 \cdot h + h^2 - 4}{h} = \frac{4 \cdot h + h^2}{h}$$

As  $h \rightarrow 0$ , the limit of the difference quotient will be:

$$\frac{4 \cdot h + h^2}{h} = \frac{h \cdot (4 + h)}{h} \rightarrow 4.$$

So, the slope of the tangent line to the function  $f(x) = x^2$  at the point  $x = 2$  is equal to 4.

### 3.3 Calculating a Formula for the Derivative Using Limits

The derivative – that is, the slope of the tangent line – of a function  $f(x)$  at a general point  $x$  is given by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Calculate a formula for the derivative of the function  $f(x) = x^2$ .

#### Solution

The difference quotient for the function  $f(x) = x^2$  at a general point  $x$  will be given by:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2 \cdot x \cdot h + h^2 - x^2}{h} = \frac{2 \cdot x \cdot h + h^2}{h}$$

The limit of this as  $h \rightarrow 0$  will be:

$$\frac{2 \cdot x \cdot h + h^2}{h} = \frac{h \cdot (2 \cdot x + h)}{h} \rightarrow 2 \cdot x.$$

So, a formula for the derivative of  $f(x)$  is:  $f'(x) = 2 \cdot x$ .

Note that if you plug  $x = 2$  into this formula, you get that the derivative when  $x = 2$  is four (just as in the previous example).

### 3.4 Summary of the Procedure for Calculating a Formula for a Derivative Using Limits

The calculation of the derivative  $f'(x)$  can be broken down into four distinct stages.

**Stage 1:** Formulate  $f(x + h)$ .

**Stage 2:** Formulate the difference quotient:  $\frac{f(x + h) - f(x)}{h}$ .

**Stage 3:** Simplify the difference quotient as much as possible.

**Stage 4:** Take the limit as  $h \rightarrow 0$ .

The result that you get will be a formula for the derivative of the function  $f(x)$ :

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

Note that the only variable in your final derivative formula should be  $x$ . (So, there should be no  $h$ 's when you are finished.)

Particular trouble spots include:

- **Formulating  $f(x + h)$  correctly.** Sometimes people forget that the “+  $h$ ” is part of the input that goes into the function and incorrectly write down an algebraic expression that puts the “+  $h$ ” on the outside of the function, i.e. they incorrectly write down  $f(x) + h$ .
- **Expanding brackets.** Sometimes people forget to multiply brackets by FOILING and incorrectly write things like:  $(x + a)^2 = x^2 + a^2$ .
- **Combining fractions.** Remember to cross-multiply.
- **Dividing by  $h$  correctly.** Remember that you can only divide by  $h$  when you have factored  $h$  out of everything that is in the numerator of the difference quotient.

So pay lots of attention to what you are doing whenever you have to do any of these operations.

### 3.5 Another Example: Calculating a Formula for the Derivative Using Limits

Use the limit definition of the derivative to find an equation for the derivative of:

$$f(x) = \frac{1}{x}.$$

## Solution

**Step 1:** Formulate  $f(x + h)$ .

$$f(x + h) = \frac{1}{x + h}$$

**Step 2:** Formulate the difference quotient:  $\frac{f(x + h) - f(x)}{h}$ .

$$\frac{f(x + h) - f(x)}{h} = \frac{\frac{1}{x + h} - \frac{1}{x}}{h}$$

**Step 3:** Simplify the difference quotient until you can cancel out the  $h$  that is present in the denominator.

$$\frac{\frac{1}{x + h} - \frac{1}{x}}{h} = \frac{\frac{x - (x + h)}{x \cdot (x + h)}}{h} = \frac{x - x - h}{x \cdot (x + h) \cdot h} = \frac{-h}{x \cdot (x + h) \cdot h}$$

**Step 4:** Take the limit as  $h \rightarrow 0$ .

$$f'(x) = \lim_{h \rightarrow 0} \frac{-h}{x \cdot (x + h) \cdot h} = \lim_{h \rightarrow 0} \frac{-1}{x \cdot (x + h)} = \frac{-1}{x^2}$$

## 4. Sketching the Graph of the Derivative Function

The derivative of a function is also a function in its own right, and so it is possible to draw the graph of the derivative function. The guiding principle for how to produce the graph of the derivative is as follows:

**The heights on the derivative graph are the slopes of tangent lines to the original function graph.**

Some other observations that can help to guide your efforts to sketch a graph of the derivative are listed below.

1. Where will the slope of the tangent line be zero on the original function graph?

- These points will be the  $x$ -intercepts of the derivative graph.

2. Where will the slope of the tangent line be positive on the original function graph?

- These are the intervals where the height of the derivative graph will be positive.

3. Where will the slope of the tangent line be negative on the original function graph?

- These are the intervals where the height of the derivative graph will be negative.

Three other points that are worth noting are:

- When the graph of the original function is concave up, the derivative graph is increasing (and vice versa).
- When the graph of the original function is concave down, the derivative graph is decreasing (and vice versa).
- (As always!) **The heights on the derivative graph are given by the slopes of tangent lines to the original function graph.**

#### 4.1 Example: Sketching Graphs of a Function and Its Derivative

Sketch a graph of the function:

$$f(x) = \frac{1}{3}x^3 - x + 2$$

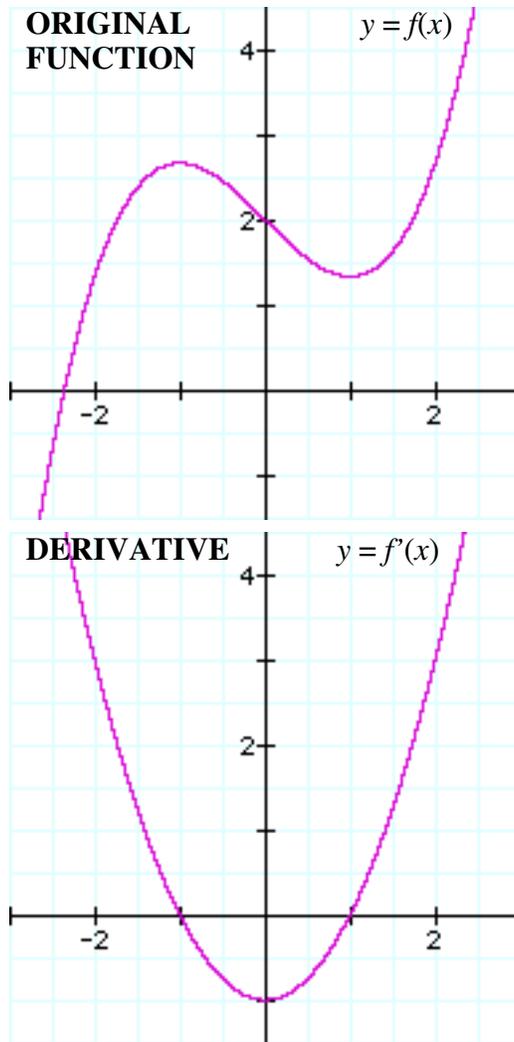
and use this original function graph to sketch a graph of the derivative.

##### **Solution**

The function and its derivative are shown on the next page. Note that the formula for the derivative of  $f(x)$  is:

$$f'(x) = x^2 - 1.$$

This is the equation for a quadratic function (which should have a parabola for a graph). The derivative graph shown on the next page does indeed resemble a parabola, which provides some additional confirmation that the derivative graph has been drawn correctly.



## 5. A Practical Interpretation of the Derivative

One interpretation that can be attached to the numerical value of the derivative at  $x = a$ , that is to the numerical value of  $f'(a)$ , is this:

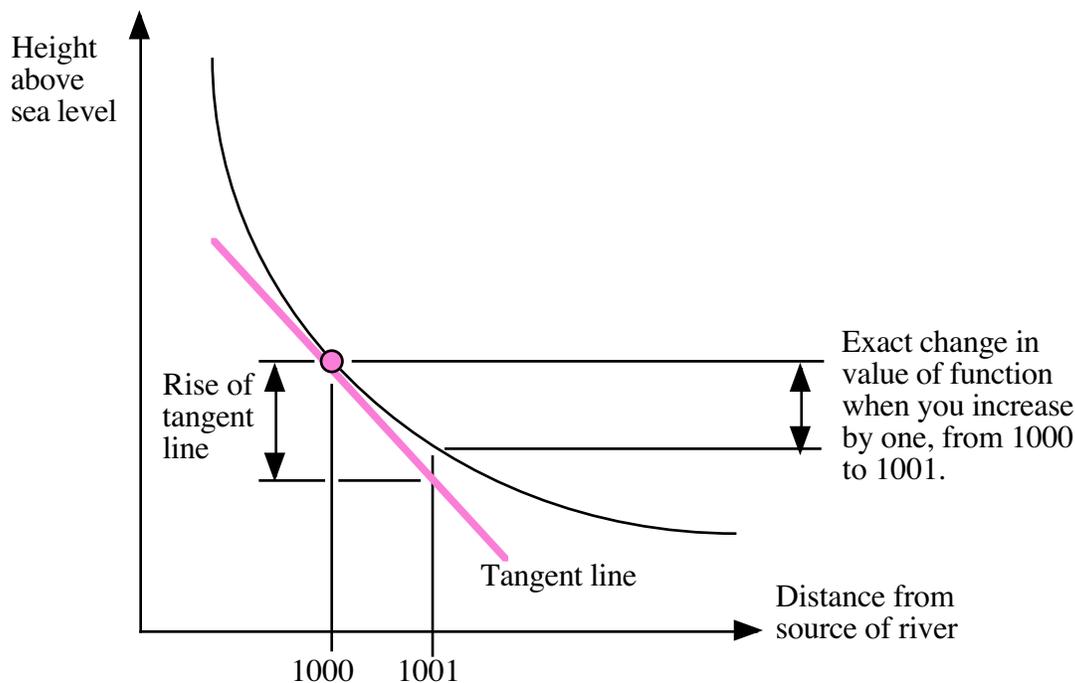
“The numerical value of  $f'(a)$  is the approximate amount that the value of the function,  $f(a)$ , changes by when  $x$  is increased by exactly one unit from  $x = a$  to  $x = a + 1$ .”

For example, if  $x$  was the distance from the source of the Mississippi river (in units of miles) and  $f(x)$  was the height above sea level (in feet), then the meaning of:

- $f'(1000) = 200$  would be that when you are 1000 miles from the course of the Mississippi, your height above sea level is 200 feet.

- $f'(1000) = -3$  would be that when you increase your distance from the source of the Mississippi from 1000 miles to 1001 miles, your height above sea level will drop by approximately 3 feet.

This **practical interpretation** for the value of the derivative can be justified graphically.



Since the “run” is equal to one, the “rise” of the tangent line is equal to the value of the derivative at  $x = 1000$ . The “rise” of the tangent line between  $x = 1000$  and  $x = 1001$  is approximately equal to the exact change of the value of the function between  $x = 1000$  to  $x = 1001$ . Hence, the value of the derivative at  $x = 1000$  is approximately equal to the change in the value of the function when you increase  $x$  by one unit from  $x = 1000$  to  $x = 1001$ .

## 6. The Second Derivative

The second derivative is a mathematical tool that enables us to easily determine the concavity of the original function without having to graph it. If  $f(x)$  is the function, then  $f'(x)$  is the derivative and  $f''(x)$  is the second derivative.

- **The second derivative of a function  $f(x)$  is the derivative of the derivative  $f'(x)$ .**

### 6.1 Examples: Calculating Derivatives and Second Derivatives

(a)  $f(x) = x^3 + x^2$       FUNCTION

Then:  $f'(x) = 3x^2 + 2x$       DERIVATIVE

And:  $f''(x) = 6x + 2$       2nd DERIVATIVE

(b)  $f(x) = 4 \cdot 7^x + x^4 + \ln(x)$  FUNCTION

Then:  $f'(x) = \ln(7) \cdot 4 \cdot 7^x + 4 \cdot x^3 + \frac{1}{x}$  DERIVATIVE

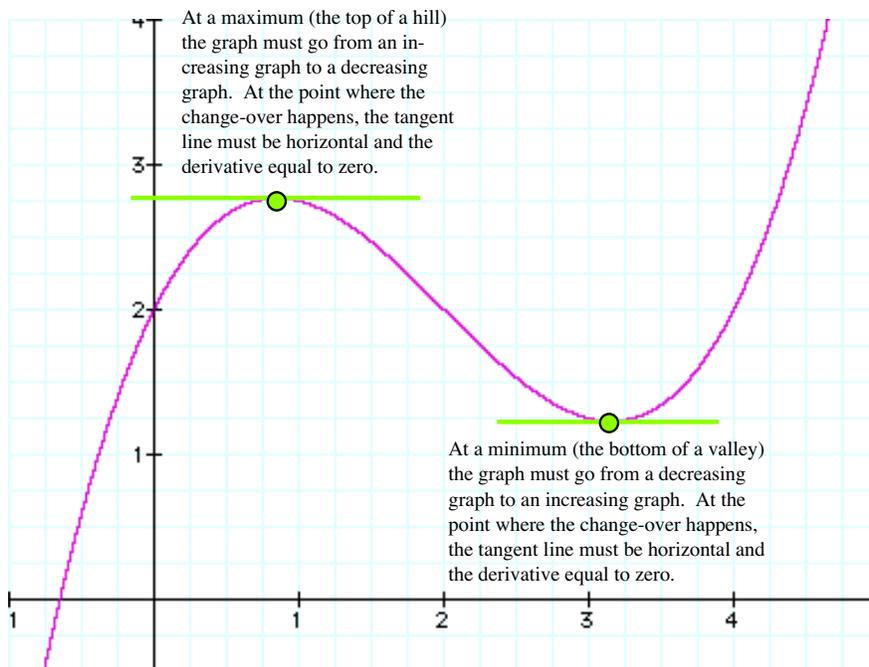
And:  $f''(x) = \ln(7) \cdot \ln(7) \cdot 4 \cdot 7^x + 12 \cdot x^2 + \frac{-1}{x^2}$  2nd DERIVATIVE

## 7. The Connections Between the Original Function, First Derivative and Second Derivative

### 7.1 Relationship between a function and its derivative

#### 7.1.1 Maximum and minimum values of the original function

These usually occur when the derivative is equal to zero.



#### 7.1.2 Places where the original function is increasing and decreasing

Behavior of original function	Sign of derivative
Increasing	+
Decreasing	-

### 7.1.3 Places where the original function is concave up and concave down

Behavior of original function	Behavior of derivative
Concave up	Increasing (may be + or -, all that is important is that the derivative is increasing)
Concave down	Decreasing (may be + or -, all that is important is that the derivative is decreasing)

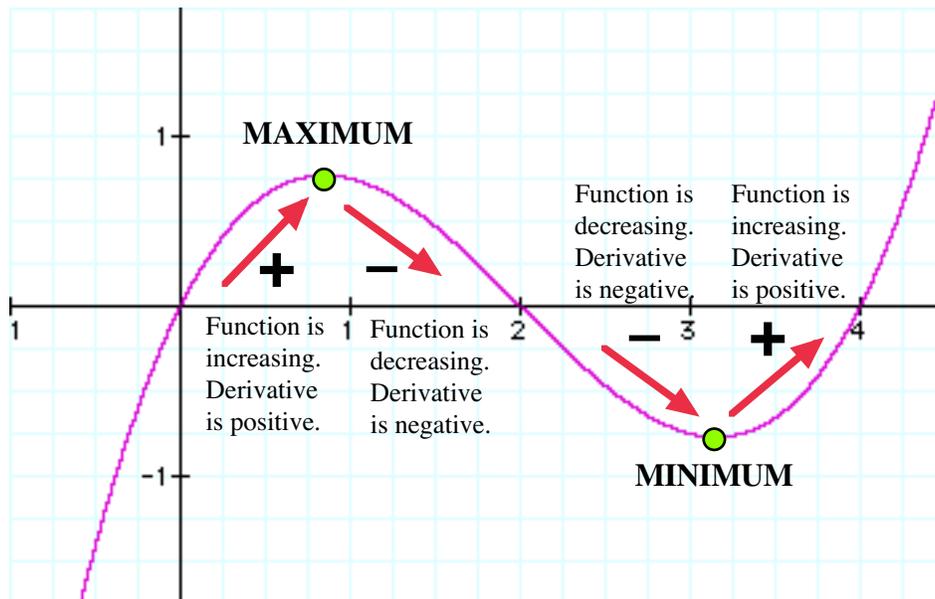
### 7.2 Using the First Derivative to Classify Critical Points as Maximums or Minimums

A **critical point** is a point at which either the derivative is equal to zero, or at which the derivative is undefined (for example, at a point where the graph of the original function has a sharp corner).

To check the type (maximum or minimum) of critical point, you can evaluate the derivative on the left and on the right of the critical point.

Sign of derivative just to the left of the critical point	Sign of derivative just to the right of the critical point	Type of critical point
+	-	Maximum
-	+	Minimum

The reason that examining the signs of the derivative on either side of a critical point can tell you about the nature of the point (i.e. whether it is a maximum or minimum) is indicated below.



### 7.3 The Sign of the Second Derivative and the Concavity of the Original Function

- The **sign of the 2nd derivative** tells you about the **concavity of the original function**.

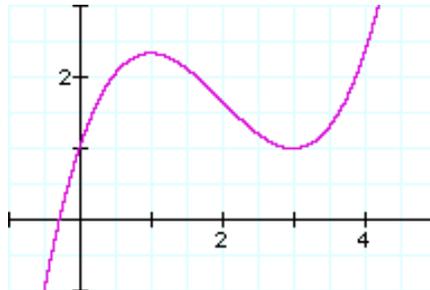
Since the second derivative is the derivative of the derivative, when the original function is concave down, the derivative is decreasing, and when the derivative is decreasing, the derivative of the derivative is negative. Overall, when the original function is concave down the second derivative is negative.

On the other hand, when the original function is concave up, the derivative is increasing. When the derivative is increasing, the derivative of the derivative is positive. Overall, when the original function is concave up, the second derivative is positive.

Second Derivative is ...	Derivative is ...	Original function is ...
Positive	Increasing	Concave up
Negative	Decreasing	Concave down
Zero	Neither increasing nor decreasing	Possibly at a point of inflection

- If the second derivative is equal to zero at a point, then the original function often has a point of inflection there - but not always. (**NOTE:**  $g(x) = x^4$  is a good counter-example.)

### 7.4 The Second Derivative and Classifying Critical Points



- Near a local maximum, the original function is concave down.
- So, near a local maximum the second derivative is negative.
- Near a local minimum, the original function is concave up.
- So, near a local minimum the second derivative is positive.

Type of Critical Point	Concavity of Original Function	Sign of Second Derivative
Local maximum	Concave down	-
Local minimum	Concave up	+