Review Notes – Calculating and Using Derivatives

Important Information:

1. According to the most recent information from the Registrar, the Xa final exam will be held from 9:15 a.m. to 12:15 p.m. on Monday, January 13 in Science Center Lecture Hall D.

2. The test will include twelve problems (each with multiple parts).

3. You will have 3 hours to complete the test.

4. You may use your calculator and one page (8" by 11.5") of notes on the test.

5. I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the test will resemble these problems in any way whatsoever.

6. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

7. Good sources of help:

   - Section leaders' office hours (posted on Xa web site).
   - Math Question Center (during the reading period).
   - Course-wide review on Friday 1/10 from 4:00-6:00 p.m. in Science Center E and Sunday 1/12 from 3:00-5:00 p.m. in Science Center A.

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1. Short Cut Rules for Calculating Derivatives

1.1 The Definitive List of All Derivative Rules that You Need to Know for Math Xa

The following list includes **all** of the derivative rules that you need to be aware of and able to use in Math Xa. If you find that you are a little rusty on finding derivatives, you might like to practice using some of the problem sets for Gateway #3 (Calculating Derivatives). You can find these problems (together with answers) at the following web site:

http://www.courses.fas.harvard.edu/~mathxa/gateway4/gateway4.html

- **Power Rule:** $\left(x^n\right)' = n \cdot x^{n-1}$

- **Sum/difference Rule:** $\left[f(x) \pm g(x)\right]' = f'(x) \pm g'(x)$

- **Constant Multiple Rule:** $\left[k \cdot f(x)\right]' = k \cdot f'(x)$

- **Derivative of an isolated constant is equal to zero.**
- **Product Rule:** $\left[f(x)\cdot g(x)\right]' = f'(x)\cdot g(x) + g'(x)\cdot f(x)$
- **Quotient Rule:** $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{\left[g(x)\right]^2}$
- **Exponential functions:** $f(x) = A\cdot B^x$ $\quad f'(x) = A\cdot\ln(B)\cdot B^x$
- **"e" to the "x":** $f(x) = e^x$ $\quad f'(x) = e^x$
- **Natural logarithm:** *Derivative of* $\ln(x) = \dfrac{1}{x}$
- **Chain Rule:** $\left[f(g(x))\right]' = f'(g(x))\cdot g'(x)$

1.2 The Product and Quotient Rules

1.2.1 The Product Rule for Derivatives

There is also a short-cut derivative rule for differentiating products of functions, called the Product Rule. The product rule is the right rule to use when you "read the algebra" of the function that you have to differentiate and see that it is made up of two other functions that have been multiplied together.

$$\left[f(x)\cdot g(x)\right]' = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

A little chant that people sometimes use to remember the product rule is:

"Derivative of the first times the second plus the derivative of the second times the first."

Examples

(a) $p(x) = (1-x)\cdot\left(1+x+x^2+x^3+x^4\right)$.

(b) $q(x) = \left(1+\sqrt{x}\right)\cdot\left(1-\sqrt{x}\right)$.

(c) $r(x) = \left(2x-x^2\right)\cdot\left(1+x^3+x^4\right)$.

Solutions

(a) $p'(x) = (-1)\cdot\left(1+x+x^2+x^3+x^4\right) + (1-x)\cdot\left(1+3x+3x^2+4x^3\right)$.

(b) $q'(x) = \frac{1}{2}x^{-1/2} \cdot \left(1 - \sqrt{x}\right) + \left(1 + \sqrt{x}\right) \cdot \left(-\frac{1}{2}x^{-1/2}\right)$.

(c) $r'(x) = (2 - 2x) \cdot \left(1 + x^3 + x^4\right) + \left(2x - x^2\right) \cdot \left(3x^2 + 4x^3\right)$.

Lastly, when differentiating products, don't forget about the Constant Multiplier rule either. It is very inefficient to differentiate a function like:

$$g(x) = k \cdot f(x)$$

using the Product rule (although still okay to do so if you want). The derivative of $g(x)$ is just k times the derivative of $f(x)$:

$$g'(x) = k \cdot f'(x)$$

1.2.1 The Quotient Rule for Derivatives

If you are trying to find the derivative of something that is the quotient of two functions, then the appropriate derivative rule to use is the Quotient Rule:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{\left[g(x)\right]^2}.$$

A chant that some people sometimes find helpful for remembering the Quotient rule is:

"Low-dee-high minus high-dee-low over low-low."

Examples

(a) $q(x) = \dfrac{1 + x^2}{1 + x}$

(b) $n(x) = \dfrac{3 \cdot \sqrt{x} + 10 \cdot x^4}{1 + x}$

(c) $m(x) = \dfrac{x + x^2}{\sqrt{x}}$

Solutions

(a) $q'(x) = \dfrac{2x \cdot (1 + x) - \left(1 + x^2\right) \cdot 1}{(1 + x)^2}$

(b) $n'(x) = \dfrac{(1 + x) \cdot \left(\frac{3}{2} \cdot x^{-1/2} + 40 \cdot x^3\right) - \left(3 \cdot \sqrt{x} + 10 \cdot x^4\right) \cdot 1}{(1 + x)^2}$

(c) $m'(x) = \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2}$

Lastly, when differentiating quotients, don't forget about the Constant Multiplier rule. It is very inefficient to differentiate a function like:

$$m(x) = \frac{f(x)}{k} = \tfrac{1}{k} \cdot f(x)$$

using the Product rule (although still okay to do so if you want). The derivative of $m(x)$ is just $1/k$ times the derivative of $f(x)$:

$$m'(x) = \frac{f'(x)}{k} = \tfrac{1}{k} \cdot f'(x).$$

1.3 The Chain Rule for Derivatives of Composite Functions

The Chain Rule is used to differentiate **composite** functions. That is, functions that have been made by using the output of one function as the input to a second function.

For example, the function

$$m(x) = \sqrt{x^2 + 2x - 1} + 4^{x^2 + 2x - 1}$$

could be thought of as made up from two separate functions:

$$f(x) = \sqrt{x} + 4^x \qquad \text{"Outside function"}$$

$$g(x) = x^2 + 2x - 1 \qquad \text{"Inside function"}$$

The function $m(x)$ is made up by starting with $g(x)$ and using the output from $g(x)$ as the input to $f(x)$ so that $m(x) = f(g(x))$.

In words, the Chain Rule for derivatives goes something like this:

i. Differentiate the **outside function**.
ii. Leave the **inside function** inside, pure and unadulterated.
iii. Now differentiate the **inside function** and multiply by this derivative.

In symbolic terms, the Chain Rule for Derivatives is stated:

$$\left[f(g(x))\right]' = f'(g(x)) \cdot g'(x).$$

1.3.1 Example 1: Using the Chain Rule to find a derivative

For example, the derivative of $m(x)$ will be:

$$m(x) = \left(x^2 + 2x - 1\right)^{\frac{1}{2}} + 4^{x^2 + 2x - 1}$$

$$m(x) = \left[\tfrac{1}{2} \cdot \left(x^2 + 2x - 1\right)^{\frac{-1}{2}} + \ln(4) \cdot 4^{x^2+2x-1}\right] \cdot (2x + 2)$$

1.3.2 Example 2: Using the Chain Rule as part of a larger derivative calculation

As another example, consider the surface area formula for the Poland Springs water bottle:

$$A = \pi r^2 + \frac{3000}{2r} + \pi r \cdot \sqrt{r^2 + \frac{9000000}{16\pi^2 r^4}}.$$

The only part of this formula that needs the Chain Rule is the square root:

$$g(r) = \left(r^2 + \frac{9000000}{16\pi^2 r^4}\right)^{\frac{1}{2}}.$$

Here the "inside function" is $p(r) = r^2 + \dfrac{9000000}{16\pi^2 r^4}$ and the outside function is $q(x) = x^{1/2}$. Using the Chain Rule to find the derivative of this square root gives:

$$g'(r) = \frac{1}{2} \cdot \left(r^2 + \frac{9000000}{16\pi^2 r^4}\right)^{-\frac{1}{2}} \cdot \left(2r - \frac{36000000}{16\pi^2 r^5}\right).$$

Therefore, the derivative of the surface area of the bottle is:

$$A' = 2\pi r - \frac{3000}{2r^2} + \pi \cdot \sqrt{r^2 + \frac{9000000}{16\pi^2 r^4}} + \pi r \cdot \frac{1}{2} \cdot \left(r^2 + \frac{9000000}{16\pi^2 r^4}\right)^{-\frac{1}{2}} \cdot \left(2r - \frac{36000000}{16\pi^2 r^5}\right).$$

1.3.1 Example 3: The derivative that showed up on your last homework

A third example is part of the Cobb-Douglas production function for Tanzania that you had to differentiate on one of your homework assignments.

$$f(x) = (13495000 - x)^{0.66}.$$

The "inside function" here is $p(x) = 13495000 - x$ and the "outside function" is $q(x) = x^{0.66}$. Using the Chain Rule, the derivative of $f(x)$ comes out to be:

$$f'(x) = 0.66 \cdot (13495000 - x)^{-0.34} \cdot (-1) = -0.66 \cdot (13495000 - x)^{-0.34}.$$

2. Locating the Maximum and Minimum Values of a Function

2.1 Points at Which the Maximum and Minimum Values of a Function May be Located

If x is restricted to an interval $[a, b]$, then the maximum and minimum values of a function $f(x)$ occur either:

- at a point where $f'(x) = 0$, or,
- at a point where it is very difficult to define $f'(x) = 0$, or,
- at one of the endpoints ($x = a$ or $x = b$) of the interval $[a, b]$.

2.2 Using the First Derivative to Classify Critical Points as Maximums or Minimums

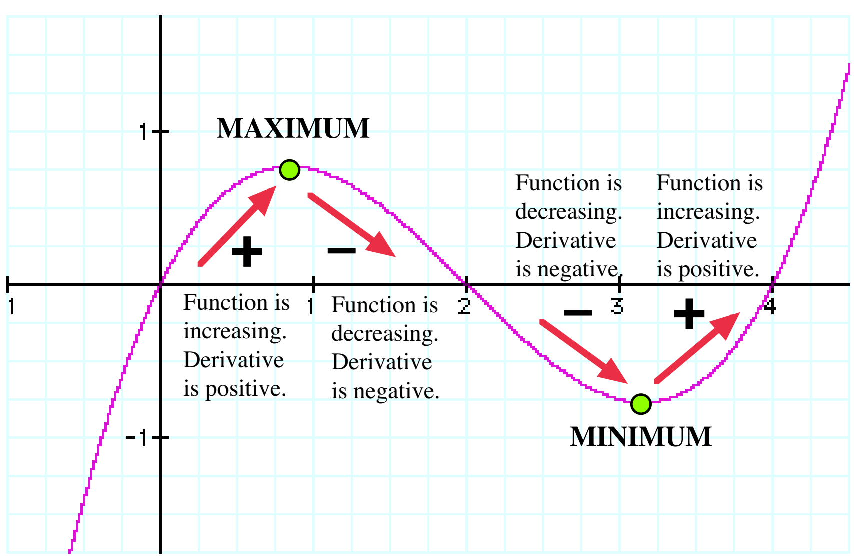
Often (but not always) the maximum and minimum values of a function will be located at the points where the derivative of the function is equal to zero.

When you have located an x-value that makes the derivative equal to zero, you often need to determine whether the point that you have found is a maximum (a "hill") or a minimum (a "valley").

You can check by evaluating the derivative at a point slightly to the left of the point, and again slightly to the right of the point. The pattern of signs of the derivative tells you whether you have found a maximum or a minimum.

Sign of derivative on left of point	Sign of derivative on right of point	Type of point
+	−	Maximum
−	+	Minimum

The rationale for this test works is shown below.



2.3 Using the Second Derivative to Classify Critical Points as Maximums or Minimums

Type of Critical Point	Concavity of Original Function	Sign of Second Derivative
Local maximum	Concave down	−
Local minimum	Concave up	+

2.4 Example

Find the maximum and minimum values of the function:

$$f(x) = x^3 - 5x + 1.$$

Solution

Step 1: **Find the derivative**

$$f'(x) = 3 \cdot x^2 - 5$$

Step 2: **Set the derivative equal to zero and solve for *x***

$$3 \cdot x^2 - 5 = 0$$

$$x = \pm\sqrt{\frac{5}{3}} \approx \pm 1.29$$

Step 3: **Use the sign of the derivative to determine which point is a maximum and which point is a minimum**

Sign of derivative just to the left of the critical point	Sign of derivative just to the right of the critical point	Type of critical point
+	−	Maximum
−	+	Minimum

In this particular case, the derivatives come out as follows:

x	$f'(x) = 3 \cdot x^2 - 5$	Interpretation
–1.3	+0.07	X=–1.29 is a maximum
–1.2	–0.68	
1.2	–0.68	X=1.29 is a minimum
1.3	+0.07	

3. Optimization

3.1 A Six-Step Checklist for Optimization Problems

Step 1: If the situation has been described graphically or verbally, try to find an equation to represent the function that you want to maximize/minimize as well as equations to represent any constraints or conditions that have to be satisfied.

Step 2: Differentiate the function and use the derivative to locate the critical points.

Step 3: Once you have located the critical points, use the derivative to decide if they are maximums, minimums or neither.

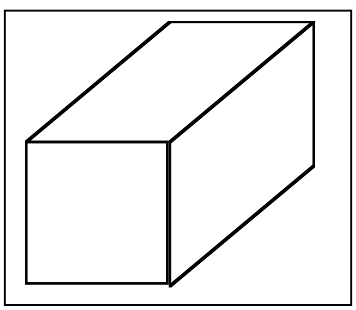
Step 4: If you have any reason to suspect that there will be points at which the derivative is not defined (e.g. absolute value, function defined in pieces) check the value of the function at those points.

Step 5: If the domain of the function is a closed interval, check the value of the functions at the end-points and compare to the value of the function at the critical points.

Step 6: Look back. Do the locations and values of the maximum/minimum seem reasonable?

3.2 Examples

3.2.1 Example 1: The biggest box that you can send through the mail



A kind relative decides to send you a package of goodies to help you through the exam period. Your relative decides to pack the goodies into a rectangular box with square ends (see diagram below). Strange postal regulations require that the sum of the length and the girth of the package not exceed 200 inches. What size box do you hope your kind relative uses?

Note: Girth is the *perimeter* of the square end of the box.

Solution

We want to maximize volume, so we need to come up with a formula for volume. If we use w to represent the width of the box and L to represent the length of the box, then:

$$V = w^2 \cdot L.$$

We can't take the derivative of this because there are too many variables (w and L). The trick is to use the postal regulation (the *constraint equation*) to eliminate one of the variables from the formula for V.

Constraint equation: $4w + L = 200$

Therefore, $L = 200 - 4w$, so that the formula for the volume becomes:

$$V = w^2 \cdot (200 - 4w).$$

The derivative is:

$$V' = 400w - 12w^2 = 4w(100 - 3w).$$

The derivative is equal to zero at $w = 0$ (which gives minimum volume) and $w = 100/3$ (which gives maximum volume). Therefore, you would want a parcel that was 33.33 inches wide and 66.66 inches long.

3.2.2 Example 2: Velocity and speed in an SDI particle beam accelerator

One of the President Reagan's more controversial programs was the Strategic Defense Initiative (popularly called "Star Wars"). One of the projects in SDI was the development of satellite-based particle beam weapons intended to shoot down Soviet nuclear missiles (see Figure 1).

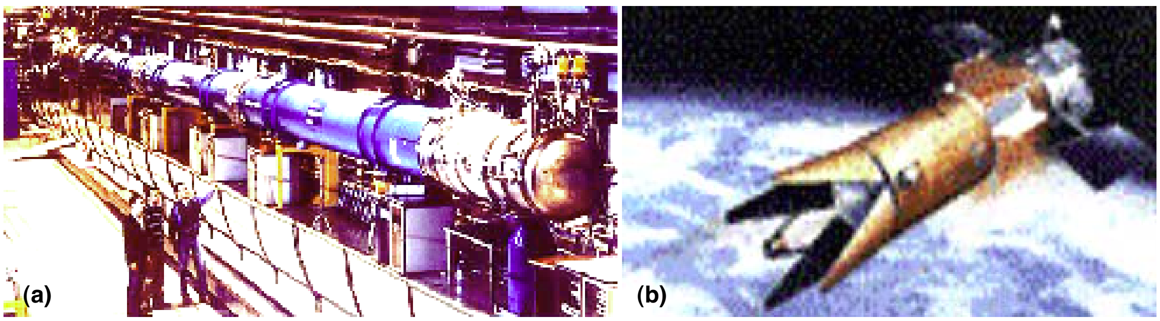


Figure 1: (a) A large, super-coducting particle accelerator similar to the specification for the SDI muon accelerator. (b) Artist's conception of the orbital SDI particle accelerator weapons platform.

One particle beam system produced, accelerated and fired muons. Muons are subatomic particles that decay with explosive results. A schematic diagram of the particle beam system is shown in Figure 2 below.

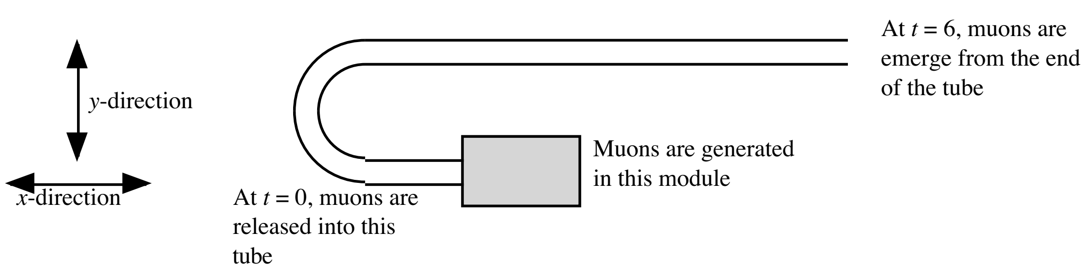


Figure 2: Schematic diagram showing how the muon accelerator was designed to fit inside the restricted space available within the SDI sattelite.

When the weapon was fired, a group of muons were produced at time t=0 and then accelerated until they left the weapon at t=6. The velocity of the muons in the x-direction was given by the following equation.

$$v(t) = t^2 + t - 6.$$

(a) When is the velocity of the muons at a maximum? Is this consistent with your expectations? When is the velocity of the muons at a minimum?

(b) Speed is the magnitude (or absolute value) of velocity. When is the speed of the muons maximized and when is the speed of the muons minimized?

Solution

(a) The derivative of the muon velocity is given by:

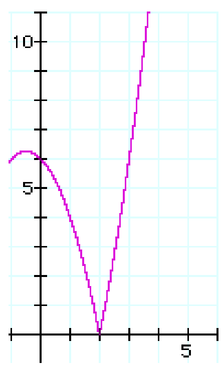
$$v'(t) = 2t + 1.$$

This is equal to zero when $t = -0.5$. This point is a minimum and is outside the range of allowable t-values for this problem. We must therefore check the end-points of the interval to determine the minimum value of the velocity. This occurs when $t = 0$ (and $v(0) = -6$).

The maximum value of velocity is achieved when $t = 6$. This is when the muons leave the tube of the accelerator, and undergo no further acceleration.

An observation to take away from this example is that when the function is restricted to a particular set of values that make sense in a particular problem – the *problem domain* of the function – the maximum and minimum values of a function may occur at the end-points of the problem domain, rather than at a point where the derivative of the function is equal to zero.

(b) If you have a look at the graph of speed versus time, it shows that the speed of the muons is minimized at $t = 2$ and maximized at $t = 6$.



Some points to take out of this are that sometimes maximum and minimum points occur where the derivative of the function is difficult to define (as is the case at $t = 2$ in this case). The derivative of a function is always hard to define at end-points and sharp-points of functions as there are usually a lot of different ways that you can "snuggle" a tangent line up against the curve at these points.

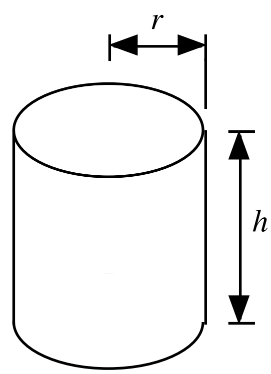
Again we can also see that when the function is restricted to a particular set of values that make sense in a particular problem – the *problem domain* of the function – the maximum and minimum values of a function may occur at the end-points of the problem domain, rather than at a point where the derivative of the function is equal to zero.

3.2.3 Example 3: What are the most environmentally friendly dimensions for a soda can?

A standard soda can is a cylinder with a circle at each end. The volume of liquid enclosed by the can is 355 ml. How high and wide should a soda can be if it is to use the least possible amount of aluminum?

Solution

In this problem we wish to minimize surface area (as the amount of aluminum in the can will be proportional to this) of the can, subject to the constraint that the volume must remain fixed at 355 ml. To do this, we will need to find formulas for the volume and surface area of a cylinder that has both ends enclosed.



Volume: $V = \pi \cdot r^2 \cdot h = 355$ (Constraint equation)

Surface Area: $A = 2 \cdot \pi \cdot r \cdot h + 2 \cdot \pi \cdot r^2$

To find the minimum value of surface area, we will differentiate *A* and then set the derivative equal to zero. There is one obstacle to doing this, however. The formula for *A* includes two variables (*r* and *h*) but we only know how to find the derivative of formulas that have one variable.

The solution to this problem is to use the volume equation to eliminate one of the variables from the surface area equation. In this particular case we will eliminate *h* from the surface area equation. Rearranging the volume equation to make *h* the subject gives the following.

$$h = \frac{355}{\pi \cdot r^2}$$

Substituting this into the surface area equation gives the following formula for *A* with *r* as the only independent variable in the equation.

$$A = 2 \cdot \pi \cdot r \cdot h + 2 \cdot \pi \cdot r^2 = 2 \cdot \pi \cdot r \cdot \left(\frac{355}{\pi \cdot r^2}\right) + 2 \cdot \pi \cdot r^2 = \frac{710}{r} + 2 \cdot \pi \cdot r^2$$

Calculating the derivative of *A* gives the following.

$$A'(r) = \frac{-710}{r^2} + 4 \cdot \pi \cdot r.$$

Next, we will set the derivative $A'(r)$ equal to zero and solve for *r*.

$$4 \cdot \pi \cdot r = \frac{710}{r^2}$$

$$r^3 = \frac{710}{4 \cdot \pi}$$

$$r = \left(\frac{710}{4 \cdot \pi}\right)^{\frac{1}{3}} \approx 3.84 \text{ cm}.$$

To verify that $r = 3.84$ (and $h = 7.67$ cm) corresponds to a minimum value of the surface area, we can check the sign of the derivative $A'(r)$ slightly to the left and slightly to the right of the critical point at $r = 3.84$. The results of these calculations (that do confirm a minimum value for surface area) are given below.

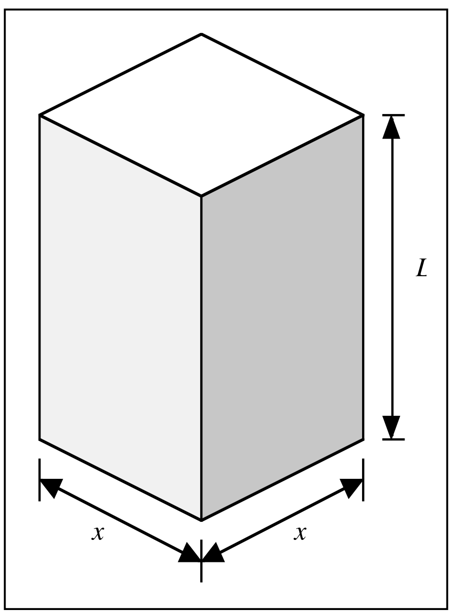
r	$A'(r)$
3.80	−1.42
3.84	0
3.90	+2.33

In contrast, a standard soda can has $r = 3.1$ and $h = 12.1$.

3.2.4 Example 4: The most environmentally friendly dimensions for a one-liter Fiji brand water bottle.

Fiji brand bottled water comes in bottles that are basically rectangular prisms with square bases. These bottles contain one liter of water. What dimensions should the bottle have in order to minimize the amount of plastic needed to manufacture it?

Solution



In this problem we wish to minimize surface area (as the amount of plastic used to manufacture the bottle will be proportional to the surface area) subject to the constraint that the volume must remain fixed at 1000 ml. In order to do this we will need to find formulas for the volume and surface area of a rectangular prism with a square base.

Volume: $V = x^2 \cdot L = 1000$ (Constraint)

Surface area: $A = 2 \cdot x^2 + 4 \cdot x \cdot L$

As in the previous example, the plan is to differentiate A and then set the derivative equal to zero. However, we also have the problem that the formula for A has two independent variables (x and L). As before, we will use the volume (constraint) equation to eliminate one of the variables (in this case, L) from the surface area formula.

Rearranging the volume formula to make L the subject of the equation gives the following.

$$L = \frac{1000}{x^2}$$

Substituting this into the surface area formula gives the following.

$$A = 2 \cdot x^2 + 4 \cdot x \cdot L = 2 \cdot x^2 + 4 \cdot x \cdot \left(\frac{1000}{x^2}\right) = 2 \cdot x^2 + \frac{4000}{x}.$$

Taking the derivative of *A* gives the following.

$$A'(x) = 4 \cdot x - \frac{4000}{x^2}.$$

Next we will set $A'(x)$ equal to zero and solve to find the value of *x*.

$$4 \cdot x = \frac{4000}{x^2}$$

$$x^3 = 1000$$

$$x = 10 \text{ cm.}$$

To show that $x = 10$cm (and $L = 10$ cm) gives a minimum value for the surface area, we can check the sign of the derivative $A'(x)$ slightly to the left and slightly to the right of the critical point at $x = 10$. The results of these calculations are summarized in the table shown below.

x	$A'(x)$
9.9	−1.21
10	0
10.1	+1.19

The actual measurements of a Fiji water bottle are $x = 7$ cm and $L = 17$ cm.

4. Implicit Differentiation

Consider the following equations that relate the independent variable (*x*) and the dependent variable (*y*).

1. $x = 3^y$

2. $x^2 + x \cdot y + y^3 = 2.$

How could you find the derivative $\frac{dy}{dx}$ from the first equation? You could re-arrange the equation so that *y* is the subject and then differentiate with respect to *x*.

Could you also use this approach if you wanted to find the derivative $\frac{dy}{dx}$ for the second equation? The second equation is much harder to rearrange to make *y* the subject.

The value of the technique of implicit differentiation technique is that it allows you to deal with situations like this - situations where you want to calculate a derivative but cannot re-arrange the equation to make *y* the subject of the equation.

Before giving some examples to illustrate the technique of implicit differentiation, note that an equation that involves x and y all mixed up together, like the following:

$$x^2 + x \cdot y + y^3 = 2,$$

implies that:

- One the of symbols (x) represents an independent variable and that the other symbol (y) represents a dependent variable so that y really represents a function with x as the input to the function.

- When taking a derivative, you have to differentiate everything with respect to x rather than differentiating things that are expressed with x with respect to x and differentiating things that are expressed with y in terms of y.

4.2 Common Steps in an Implicit Differentiation Problem

1. Differentiate each individual term in the expression, using whatever differentiation rule (product, quotient, chain) is called for. Remember that every time you differentiate something that involves y, the Chain Rule requires that you multiply by a factor of $\frac{dy}{dx}$.

2. Move every term that includes $\frac{dy}{dx}$ to one side of the equation.

3. Move every term that does not include the derivative to the other side of the equation.

4. Factor out the $\frac{dy}{dx}$.

5. Divide by the bracket that you get to make $\frac{dy}{dx}$ the subject of the equation.

4.3 Example

Find a formula for $\frac{dy}{dx}$ when x and y are related by the following equation:

$$x^2 + x \cdot y + y^3 = 2.$$

Solution

First, we will go through and differentiate everything. Remember that x is the independent variable and y is the dependent variable.

$$2x + 1 \cdot y + x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = 0$$

Now, we rearrange to get everything involving the derivative on one side of the equation and every term that does not involve the derivative on the other side of the equation.

$$x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = -2x - y$$

Next, factor out the derivative.

$$\left[x + 3y^2\right] \cdot \frac{dy}{dx} = -2x - y$$

Finally, divide by the bracket to make the derivative the subject of the equation.

$$\frac{dy}{dx} = \frac{-2x - y}{x + 3y^2}$$

4.4 Example

Find an equation for $\frac{dy}{dx}$ starting:

$$e^y + \frac{x}{y} = 10\,.$$

Solution

First, we will go through and differentiate everything. Remember that x is the independent variable and y is the dependent variable.

$$e^y \cdot \frac{dy}{dx} + \frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2} = 0$$

Now, we rearrange to get everything involving the derivative on one side of the equation and every term that does not involve the derivative on the other side of the equation.

$$e^y \cdot \frac{dy}{dx} + \frac{-x}{y^2} \cdot \frac{dy}{dx} = \frac{-1}{y}$$

Next, factor out the derivative.

$$\left[e^y + \frac{-x}{y^2}\right] \cdot \frac{dy}{dx} = \frac{-1}{y}$$

Finally, divide by the bracket to make the derivative the subject of the equation.

$$\frac{dy}{dx} = \frac{\frac{-1}{y}}{e^y + \frac{-x}{y^2}}.$$

5. Related Rates

The objective of a related rates problem is to find the numerical value of one derivative, using the numerical value of a different (but related) derivative.

1. Read the problem thoroughly and carefully.

2. Identify the problem that you have to solve (or quantity that you have to calculate).

3. Identify all of the quantities that are described in the problem.

4. Determine how the quantities are related.

- A carefully drawn diagram can be a big help here.

- The Theorem of Pythagoras, The Principle of Similar Triangles and Volume/Area formulas are all very useful tools for finding relationships between the two quantities.

5. Differentiate using the Chain Rule.

6. Solve for the quantity that you want to calculate.

5.1 Example: Sabotaging the Ladder of Divine Ascent

Figure 3(a) (see over) show the 15^{th} century painting "Ladder of Divine Ascent" by St. John Climacus. The original hangs in the Lenin Library, Moscow. In this painting, the faithful are shown ascending to paradise on a ladder of questionable stability. The ascent of the faithful is complicated by a gang of devils who are trying to knock the faithful off the ladder with bows, spears and tridents.

It doesn't take much of a theologian to see that if their goal is to prevent the faithful from reaching paradise, then the devils are pursuing a very inefficient strategy – picking off the faithful one by one. The devils would be far more effective if they simply pushed the base of the ladder, causing it to collapse, and taking all of the faithful down simultaneously.

Your ultimate goal on this handout is to calculate how fast the individual who almost made it into paradise is falling when the top of the ladder is only ten feet off the ground.

(a) In terms of the quantities defined in Figure 3(b) and their derivatives, what quantity or derivative have you been asked to calculate?

(b) What derivative is given in Figure 3(b)? What quantity in Figure 3(b) is this the derivative of?

(c) What is the relationship between x and y shown in Figure 3(b)?

(d) What is the relationship between the derivatives $\frac{dx}{dt}$ and $\frac{dx}{dt}$?

(e) How fast is the individual who almost made it into paradise falling when the top of the ladder is ten feet off the ground?

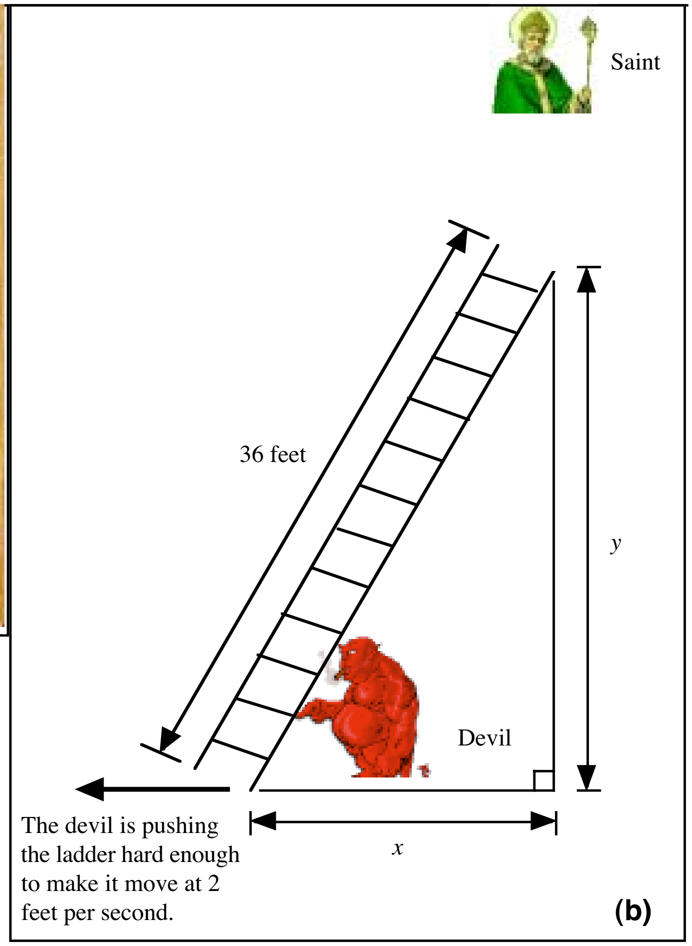


Figure 3: (a) "Ladder of Divine Ascent" painted by Saint John Climacus in the Fifteenth Century. (b) A more efficient strategy to prevent the faithful from reaching paradise. The devil pushes the bottom of the rickety ladder, causing all of the faithful to plummet to their doom, presumably to spend eternity in the company of Beelezbub and his hellish implements of torture and damnation.

Solution

(a) You are being asked to calculate $\frac{dx}{dt}$ at the particular moment of time when $y = 10$.

(b) The derivative is the speed that you are given (2 feet per second) brought about by the devil pushing on the ladder. In symbols this is $\frac{dx}{dt} = 2$.

(c) The Theorem of Pythagoras gives that: $x^2 + y^2 = 36^2$.

(d) $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$

(e) When $y = 10$, $x = 34.58$. Therefore:

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-34.58}{10} \times 2 = -6.92 \text{ feet per second.}$$

6. Slope Fields

It is possible to define a function by giving just two pieces of information:

1. The rate of change of the function.
2. The coordinates of one point that lies on the graph of the function.

When a function is specified in this way, you will not usually be given an explicit formula with which to work out exact values of the function. Furthermore, it is not at all easy (and sometimes simply impossible) to work backwards from the rate of change to construct a formula for the function.

A slope field is a tool that enables you to produce a fairly accurate picture of the graph of a function that has been defined by a rate of change. Once you have the graph, you can use it to approximate values of the function. This is not perfect – after all, you will only be approximating the values of the function – but it is usually better than nothing.

6.1 Sketching a Slope Field from an Equation for the Rate of Change

To draw the slope field starting with an equation for the rate of change:

1. At each point on the grid, calculate the value of the derivative.

2. At each point where you calculate the derivative, draw a little line segment going through that point with slope given by the value of the derivative.

3. Locate the initial value on the "grid." This is the starting point for your curve.

4. Sketch in the curve using the little line segments as "guide lines."

The curve that you have drawn actually represents:

- The differential equation describes a relationship between two quantities (an independent variable, e.g. **time**, and a dependent variable, e.g. **number of drug users**).

- The curve that you sketch (using the slope field as their guide) is a graph of the relation defined by the differential equation (in this case it was a graph of **number of drug users** versus **time**).

- Slope fields give you a way of sketching the graph of a function even if all you are told is the rate of change of the function.

6.2 Example: Sketching the slope field and graph for a function defined by its rate of change

In 1991, the number[1] of high school seniors who had used any kind of illicit drug [2] was about 29,400[3]. If we let T represent the number of years since 1990 and $N(T)$ represent the number of high school seniors who have used some illicit drug, then the derivative is given by the equation[4]:

---

[1] Here the "number of illicit drug users" means: If you took a random sample of 100,000 high school seniors from across the United States, the number of seniors in this group who had used illicit drugs at some point in their lives would be the "number of illicit drug users."

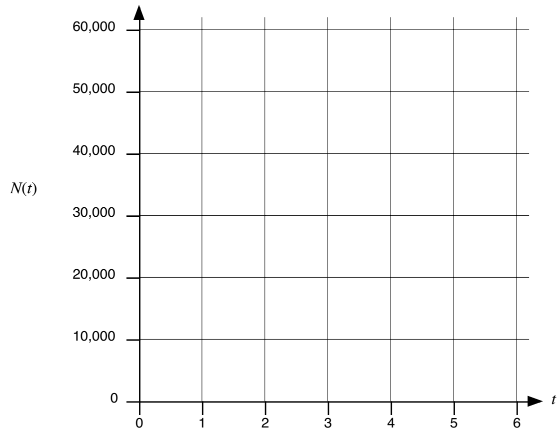
[2] This includes marijuana/hashish, inhalants, nitrites, LSD, hallucinogens other than LSD, PCP, Ecstasy, Cocaine (powder and crack), heroin, amphetamines, barbituates, tranquilizers, rohypnol, GHB, ketamine, but **excludes** alcohol, tobacco and steroids.

[3] Source: US Department of Health and Human Services, National Institute on Drug Abuse, Monitoring the Future Study, 2001.

[4] The equation for the derivative is obtained from data recorded in the Monitoring the Future Study, 2001.

$$N'(T) = -0.65 \cdot (N(T) - 43{,}200).$$

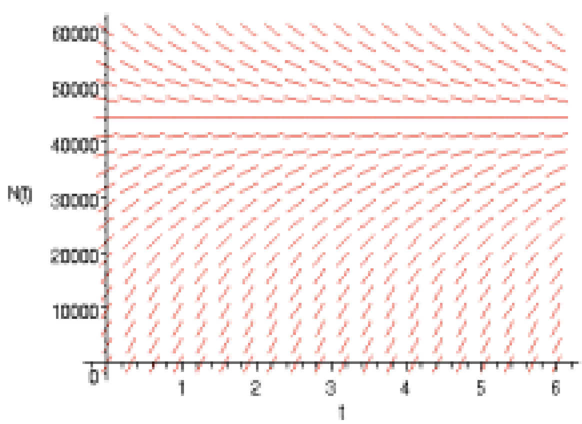
1. Use the axes provided below to draw a slope field using the equation for $N'(t)$ given above.



2. According to the U.S. Department of Health and Human Services[5], in 1991 there were about 29,400 high school seniors who had used some kind of illicit drug during their lives. Use this information (together with your slope field from Question 1) to sketch a graph showing the number of illicit drug-using high school seniors as a function of time.

Solution

1. The slope field for the equation: $N'(T) = -0.65 \cdot (N(T) - 43{,}200)$.
is given below. (**NOTE:** Clearly, the slope field given here was drawn with the aid of a specialized computer program[6]. So long as your slope field resembles this picture in its qualitative features, your answer will be perfectly acceptable.)



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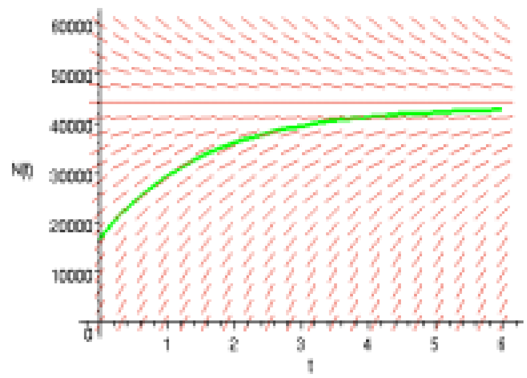
[5] Source: US Department of Health and Human Services, National Institute on Drug Abuse, Monitoring the Future Study, 2001.
[6] The program used for this picture was the computer algebra system Maple 6. If you are familiar with Maple 6, the specific commands used to produce the picture were:

```
[> with(DEtools):
[> dfieldplot(diff(N(t),t)=-0.65*(N(t)-43200),N(t),t=0..6,N=0..60000,
   arrows=line);
```

2. The solution curve that passes through the point (1, 29400) is shown (in green) on the slope field given below. (**NOTE:** Again, this graph was produced with the aid of a computer program[7]. So long as your graph is consistent with the qualitative features of the picture shown below, your graph and slope field will be perfectly acceptable.)



[7] The Maple 6 program was also used to produce this plot. The commands for this plot were:

```
[> with(DEtools):
[> DEplot(diff(N(t),t)=-0.65*(N(t)-43200),N(t),t=0..6,N=0..60000,
   [[N(1)=29400]],linecolor=green,arrows=line);
```