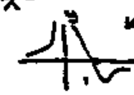
Answers to Some Review Problems

for the Second Exam

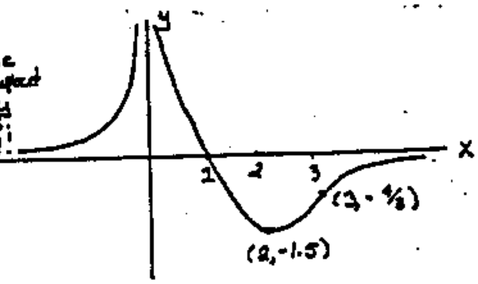
(1)

$y = 6/x^2 - 6/x = 6x^{-2} - 6x^{-1} = \frac{6}{x^2} - \frac{6x}{x^2} = \frac{6(1-x)}{x^2}$

sign of y: $+$ | 0 | $+$ | 1 | $-$ Roughly



this is the picture w/out using any calculus: Not bad!



x-intercept: $x=1$; horizontal asympt. at $y=0$

a) $y' = -12x^{-3} + 6x^{-2} = \frac{-12}{x^3} + \frac{6x}{x^3} = \frac{+6(-2+x)}{x^3}$

critical pts: $x=2$, $\cancel{x=0}$ ← Not in the domain

sign of y': $+$ | 0 | $-$ | 2 | $+$ local min at $x=2$; local min value: $\frac{6-12}{4} = \frac{-6}{4} = -1.5$

No absolute max

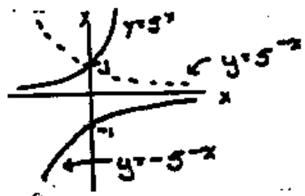
Absolute (global) min at $x=2$ (also a local min)

b) Pts of inflection? $y'' = 36x^{-4} - 12x^{-3} = \frac{36}{x^4} - \frac{12x}{x^4} = \frac{36-12x}{x^4} = \frac{12(3-x)}{x^4}$

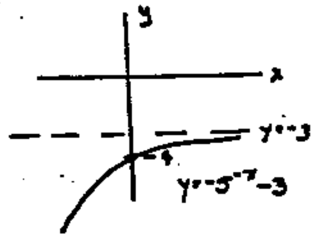
sign of y'': $+$ (y concave up) | 0 | $+$ (y concave up) | 3 | $-$ (y concave down)   pt. of inflection: $x=3$ ($y = \frac{6(1-3)}{9} = \frac{-4}{3}$)

General Advice: I'd suggest first looking at $y = \frac{6(1-x)}{x^2}$ and making a crude graph just knowing the vertical & horizontal asymptotes & pos/neg. From this you know you should get some min for $x>1$. you know y'' is eventually neg. for large x (& pos. for very neg. x). You know an awful lot! You use y' & y'' for fine tuning only.

(2)



so $y = -5^{-x} - 3 \Rightarrow$



y-int. $y=-4$
horiz. asympt. $y=-3$

(3)

Maximize volume:

$V = \underbrace{\pi r^2 h}_{\text{vol of cyl.}} + \underbrace{\frac{4}{3}\pi r^3}_{\text{vol of sphere}}$

We must relate h & r: Surface area fixed: $4\pi r^2 + 2\pi r h = F$

$h = \frac{F - 4\pi r^2}{2\pi r}$

$\text{Vol} = \pi r^2 \left(\frac{F - 4\pi r^2}{2\pi r}\right) + \frac{4}{3}\pi r^3$

$= \frac{r}{2}(F - 4\pi r^2) + \frac{4}{3}\pi r^3 = \frac{F}{2} r - 2\pi r^2 + \frac{4}{3}\pi r^3 = \frac{F}{2} r - \frac{2}{3}\pi r^3$: this is a function of 1 variable: r

$V'(r) = \frac{F}{2} - 2\pi r^2 = 0$

$2\pi r^2 = \frac{F}{2} \Rightarrow r^2 = \frac{F}{4\pi} \Rightarrow r = \pm\sqrt{\frac{F}{4\pi}}$ but $r > 0 \Rightarrow \boxed{r = \frac{1}{2}\sqrt{\frac{F}{\pi}}}$

$V''(r) = -4\pi r < 0$ for r positive - so $r = \frac{1}{2}\sqrt{\frac{F}{\pi}}$ is a local max.

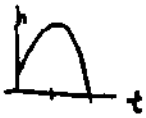
What's h? $h = \frac{F - 4\pi r^2}{2\pi r} = \frac{F - 4\pi \frac{F}{4\pi}}{2\pi\sqrt{\frac{F}{4\pi}}} = \frac{F - F}{2\pi\sqrt{\frac{F}{4\pi}}} = 0$ so $h = 0$.

What does this mean? Make a spherical capsule. r is as big as it can be w/ F fixed.

(4)

a) $h(t) = -16t^2 + 40t + 3$

$h' = -32t + 40 = 0$



$t = \frac{40}{32} = \frac{10}{8} = \frac{5}{4} = 1.25$ Max at $t = 1.25$

b) $h = -16t^2 + 40t + 3 = 0$

$$t = \frac{-40 \pm \sqrt{1600 + 4(48)}}{-32} = \frac{-40 \pm \sqrt{1792}}{-32} \approx \frac{-40 \pm 42.332}{-32}$$

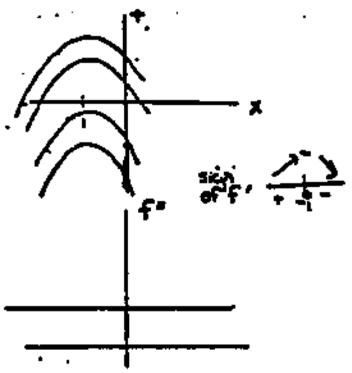
we want $t > 0$, so $t = \frac{-40 - \sqrt{1792}}{-32} \approx 2.57$

c) We want $|h'| = |-32t + 40|$ max - this will happen when the object hits the ground. at $t = \frac{+40 + \sqrt{1792}}{32} \approx 2.57$

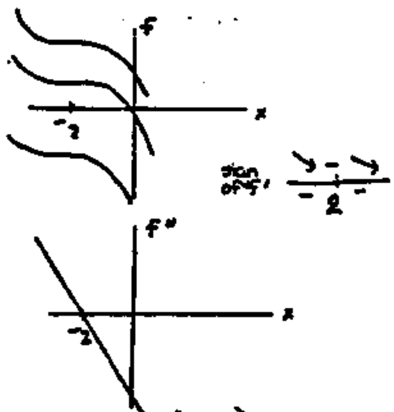
d) acceleration $= h'' = -32$: its constant. This makes sense - its the acceleration due to gravity.

(5)

a)

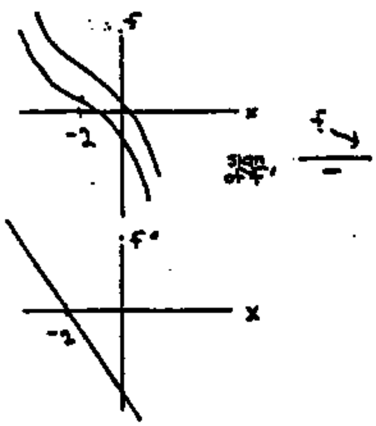


b)

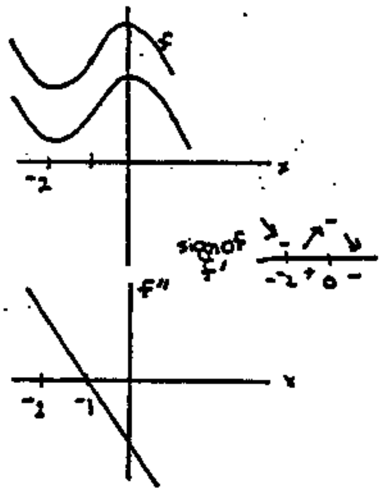


(or something similar)

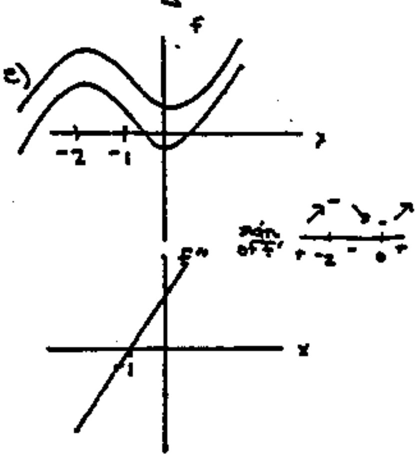
c)



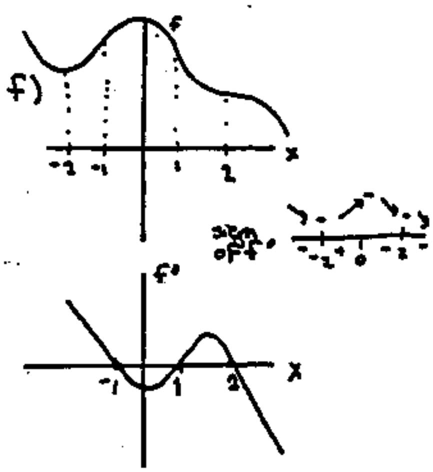
d)



e)

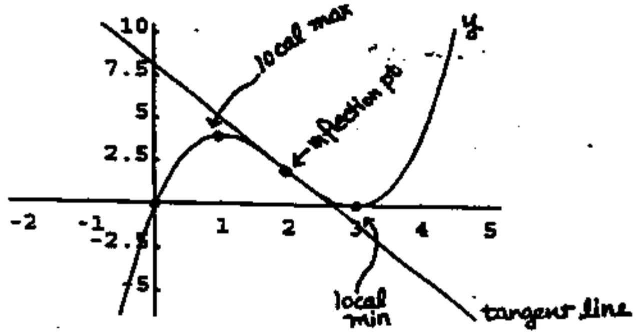


f)



General Advice: When you see the graph of f' make a "sign of f' number line". It forces you to look at where f' is positive - where f' is negative -

6 a)



$$\begin{aligned} y &= x^3 - 6x^2 + 9x \\ y' &= 3x^2 - 12x + 9 \end{aligned}$$

To find critical points set $y' = 0$:

$$y' = 3x^2 - 12x + 9 = 0$$

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x-3)(x-1) &= 0 \end{aligned}$$

$$x = 3 \qquad x = 1$$

Using the second derivative test we can determine whether each critical point is a local max or a local min:

$$y'' = 6x - 12$$

$$y''(1) = 6 - 12 = -6 < 0$$

Therefore, $x = 1$ is a local maximum.

$$y''(3) = 18 - 12 = 6 > 0$$

Therefore, $x = 3$ is a local minimum.

To find the points of inflection:

$$y'' = 6x - 12 = 0$$

$$x = 2$$

Since $y'' > 0$ for $x > 2$ and $y'' < 0$ for $x < 2$, y has an inflection point at $x = 2$.

local maximum point: $x = 1$
local minimum point: $x = 3$
point of inflection: $x = 2$

b) Slope at the point of inflection $= 3(2)^2 - 12(2) + 9 = -3$.
Therefore, the equation of the line must be in the form:

$$y = -3x + b.$$

Since $y(2) = 2^3 - 6(2)^2 + (9)(2) = 2$, the line must pass through (2,2).
So,

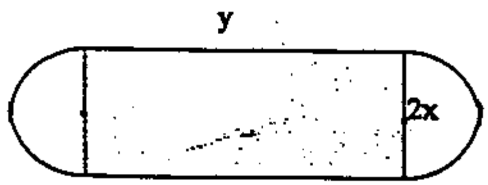
$$\begin{aligned} 2 &= (-3)(2) + b \\ b &= 8 \end{aligned}$$

Therefore, the equation of the tangent line at the point of inflection is:

$$y = -3x + 8.$$

[Tangent line is graphed on the figure above]

7 Let y be the width of the rectangular part of the racetrack and let x be the radius of the semicircle on each end in the figure below.



The area that we need to maximize is given by

$$A = 2x \cdot y.$$

We know that the perimeter of the entire racetrack is a constant P. So,

$$P = 2y + 2\pi x$$

Solving for y:

$$2y = P - 2\pi x$$

$$y = P/2 - \pi x$$

Substituting y into the area function:

$$A = 2x(P/2 - \pi x) = Px - 2\pi x^2$$

To determine critical points, find the roots of the derivative:

$$A' = P - 4\pi x = 0$$

So,

$$x = \frac{P}{4\pi}$$

We should verify that this critical point is at a max using the second derivative test:

$$A'' = -4\pi < 0 \qquad \text{for all } x$$

Therefore, $x = \frac{P}{4\pi}$ is a max.
Thus, the dimensions that maximize the rectangle are:

$$2x = \frac{P}{2\pi}$$

$$y = P/2 - \pi\left(\frac{P}{4\pi}\right) = P/2 - P/4 = P/4.$$

If the perimeter is 7 furlongs, then the dimensions will be

$$2x = \frac{P}{2\pi} = \frac{7}{2\pi} \text{ furlongs} = \frac{7}{16\pi} \text{miles}$$

$$y = P/4 = 7/4 \text{ furlongs} = 7/32 \text{ miles}$$

Therefore, the maximum area in square miles $= \frac{7}{16\pi} \cdot \frac{7}{32} = \frac{49}{512\pi}$ square miles.

8 The following are true statements:
(b) f has a local maximum point at $x = 2$.
(g) possibly f has a global maximum at $x = -2$, we don't have enough information to say for sure.

9

a) $G(t) = 800(1.06)^t$

10

$V = \pi r^2 h \quad 108 = 2\pi r + h$

Get V in terms of one variable:

$V = \pi r^2(108 - 2\pi r)$

$= 108\pi r^2 - 2\pi^2 r^3$

$\frac{dV}{dr} = 216\pi r - 6\pi^2 r^2$

$= 6\pi r(36 - \pi r)$

$\frac{dV}{dr} = 0 \Rightarrow r = 0, \; r = \frac{36}{\pi}$

f: ↗ ↘

f': (endpt) 0 $+$ $\frac{36}{\pi}$ $-$

So, $r = \frac{36}{\pi}$ is local min.

$h = 108 - 2\pi r = 108 - 2\pi\left(\frac{36}{\pi}\right) = 36$

$\boxed{\text{radius} = \frac{36}{\pi}, \ \text{height} = 36}$

11

a) $f(x-2)$: $\boxed{\#6}$ shift rt. 2 units

b) $-f(x)+2$: $\boxed{\#4}$ flip over x-axis, move ↑ 2

c) $f(2x)$: $\boxed{\#5}$ squished horizontally by a factor of 2

d) $f(\frac{x}{2})$ $\boxed{\#7}$ stretched horizontally by a factor of 2.

Common Errors $f(2x)$: replacing x by $2x$ squishes, & does NOT stretch. x can be ½ as big as it used to be to get the same y-value.

Try values!! Eg. for a) try $x=2$: $f(2-2)=f(0)=0$ so a) can't correspond to #1 !