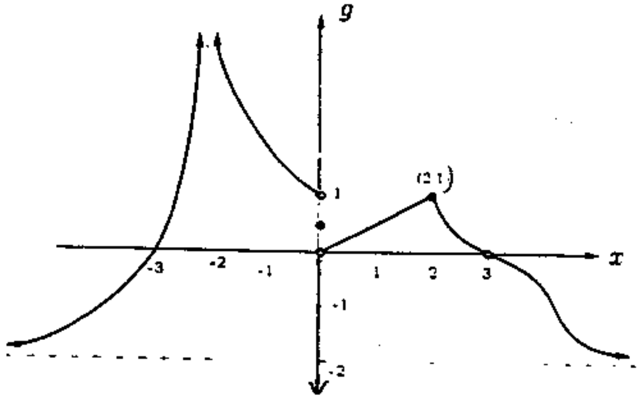
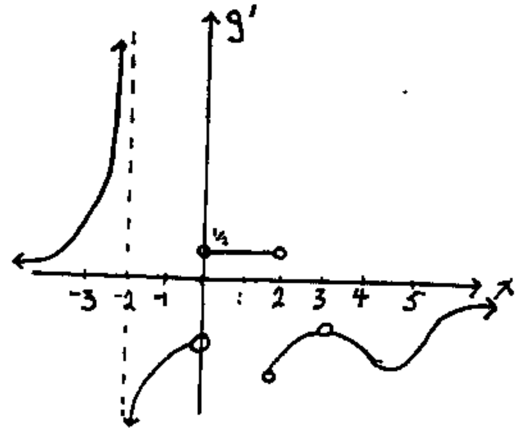
Annotated Solutions to the Second Xa Exam
12/3/98

(#2)





(a) $\lim_{x\to -2^-} g(x) = \infty$ (and similarly, $\lim_{x\to -2^-} g(x) = \infty$, so $\lim_{x\to -2} g(x) = \infty$ even though -2 is not in the domain of g.)

(b) $\lim_{x\to 0^-} g(x) = 1$ (and $\lim_{x\to 0^+} g(x) = 0$)

(c) $\lim_{x\to 0} g(x)$ is undefined. The left- and right-hand limits are unequal.

(d) $\lim_{x\to \infty} g(x) = -2$ (similarly, $\lim_{x\to -\infty} g(x) = -2$)

(e) $\lim_{x\to 2} g(x) = 1$ ($\lim_{x\to 2^+} g(x) = \lim_{x\to 2^-} g(x) = 1$, so $\lim_{x\to 2} g(x) = 1$. Even if $g(2)$ were defined to be π, for instance, $\lim_{x\to 2} g(x) = 1$.

(f) $\lim_{x\to 3} g(x) = 0$ ($\lim_{x\to 3^+} g(x) = \lim_{x\to 3^-} g(x) = 0$; so $\lim_{x\to 3} g(x) = 0$)

e the graph of g' above

(g) $\lim_{x\to \infty} g'(x) = 0$

(h) $\lim_{x\to 2^-} g'(x) = \frac{1}{2}$ for all $x \in (0,2)$, $g'(x) = \frac{1}{2}$: that's the slope of the line joining $(0,0)$ and $(2,1)$

(i) $\lim_{x\to 2} g'(x)$ is undefined. $\lim_{x\to 2^-} g'(x) = \frac{1}{2}$ but $\lim_{x\to 2^+} g'(x)$ is some negative #, so $\lim_{x\to 2^-} g'(x) \neq \lim_{x\to 2^+} g'(x)$

(j) $\lim_{x\to -2} g'(x)$ is undefined $\lim_{x\to -2^+} g'(x) = -\infty$ while $\lim_{x\to -2^-} g'(x) = +\infty$
$\lim_{x\to -2^+} g'(x) \neq \lim_{x\to -2^-} g'(x)$; therefore $\lim_{x\to -2} g'(x)$ is undefined.

(k) At what points in the domain of g is $g'(x)$ undefined?

$g'(x)$ is undefined at $\boxed{x=0}$ and $\boxed{x=2}$. At both of these points the function g is not locally linear and the limit definition of derivative can't be applied since the relevant limit doesn't exist.

While $g'(x)$ fails to exist at $x=-2$ and $x=3$, these points are not in the domain of g

#3) (a) $f'(x) = 12x^3 - 12\pi x^2$. Critical points where $f'(x) = 0$ (since f' is defined everywhere, and there are no endpoints for the domain).
since $f'(x) = x^2(12x - 12\pi)$, then $x = 0$ and $x = \pi$

(b) Look among critical points $x=0$ and π for extrema. Check sign of $f'(x)$: $\underline{-\ 0\ -\ \pi\ +}$, so local min at $x = \pi$, looks like point of inflection at $x=0$, but not a min or max

local minima: $\underline{x = \pi}$

local maxima: $\underline{\text{none}}$

Does f have an absolute minimum? $\underline{\text{yes}}$

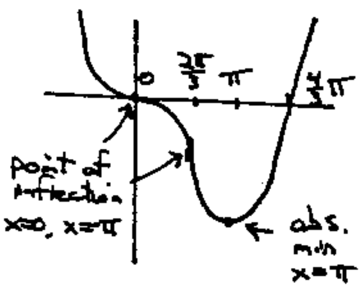
If so... coordinates are $(\pi, f(\pi)) = \underline{(\pi, -\pi^4)}$

Does f have an absolute maximum? $\underline{\text{no}}$

(c) $f''(x) = 36x^2 - 24\pi x$. $f''(x) = 0$ when $x = 0$ or $x = \frac{2}{3}\pi$
Check that sign of $f''(x)$ changes at $x=0$ and $x = \frac{2}{3}\pi$
$f''(x)$: $\underline{+\ 0\ -\ \frac{2}{3}\pi\ +}$. Yes, so both are points of inflection, and since these are the only roots of $f''(x)$ then there aren't any others.

(d) $f'(0) = 12(0)^3 - 12\pi(0)^2 = 0$, $f'(\frac{2}{3}\pi) = 12\left(\frac{2\pi}{3}\right)^3 - 12\pi\left(\frac{2\pi}{3}\right)^2 = \frac{-16\pi^3}{9}$

(e)



need to find
roots of $f(x) = 3x^4 - 4\pi x^3 = x^3(3 - 4\pi)$
so at $x=0$, $x = \frac{4}{3}\pi$

(f) $f(x-\pi) + 4$ is the function you get by shifting the graph shown above to the right by π units along x-axis, and shifted up 4 units along y-axis (vertically). Since the absolute minimum is at $(\pi, -\pi^4)$ and π^4 is much greater than 4, then $f(x-\pi)+4$ will still have two roots, or two solutions to $f(x-\pi) + 4 = 0$.

4. Since $f(x)$ is a 3^{rd} degree polynomial with a negative leading coefficient, we know that

- f is continuous and differentiable on $(-\infty, \infty)$ (All polynomials are)
- f looks like  or  (since f is 3^{rd} degree and $a<0$)

a. f is always negative. This is FALSE. $\lim_{x\to-\infty} f(x) = +\infty$

b. f has an absolute max but not an absolute min. This is FALSE. f has neither. $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to\infty} f(x) = -$

c. f is continuous on $(-\infty, \infty)$. This is true.

d. $\lim_{x\to\infty} f'(x) = \lim_{x\to-\infty} f'(x)$. True: $\lim_{x\to+\infty} f'(x) = -\infty$ & $\lim_{x\to-\infty} f'(x) = -\infty$

OR: $f'(x) = 3ax^2 + 2bx + c$ where $a<0$.

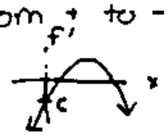
so $f'(x)$ looks like 

e. $\lim_{x\to-\infty} f(x) = -\infty$. This is FALSE.

f. f has at most two turning pts. Yes, f' is a quadratic, so it can change sign from + to − or vice-versa at most twice.

g. f is positive and decreasing at $x=0$

$$\begin{cases} f(0)=d>0 \Rightarrow f \text{ positive at } 0 \\ f'(0)=c<0 \Rightarrow f \text{ decreasing at } 0 \end{cases}$$

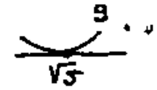


h. f is differentiable everywhere on $(-\infty, \infty)$.

i. $\lim_{x\to-\infty} |f(x)|^2 = -\infty$. False! $|f(x)|^2 \geq 0$ for ANY function f.

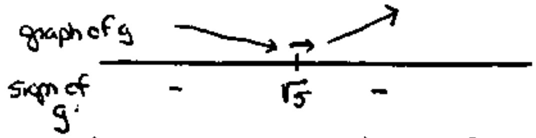
6. a. The second derivative test says

if $\underbrace{g'(\sqrt{5})=0}_{\text{g has a horizontal tangent line at } x=\sqrt{5}}$ and $\underbrace{g''(\sqrt{5})>0}_{\text{g is concave up at } x=\sqrt{5}}$ then g has a local minimum at $x=\sqrt{5}$



The first derivative test says

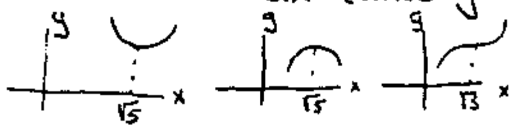
if $g'<0$ for all $x<\sqrt{5}$ and $g'>0$ for all $x>\sqrt{5}$ then $x=\sqrt{5}$ is a local min



so (i) and (v) guarantee a local minimum at $x=\sqrt{5}$

(ii) guarantees a local max (since $g''(5)<0 \Rightarrow g$ is concave down at $x=\sqrt{5}$

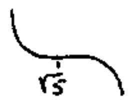
(iii)



inconclusive

(Think of $g(x)=-(x-\sqrt{5})^4+1$ for example; here there's a max at $x=\sqrt{5}$.)

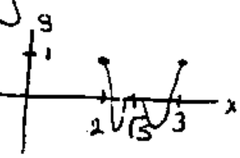
(iv)



This is a possibility, not necessarily a min at $x=\sqrt{5}$

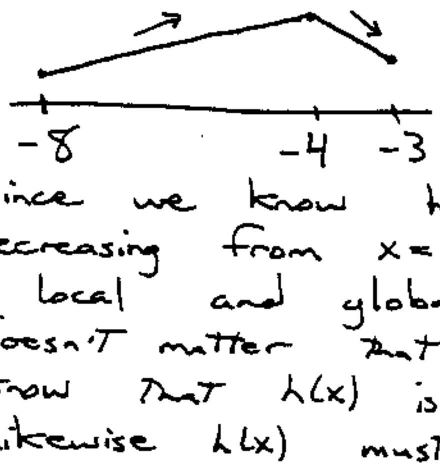
(vi) This is inconclusive! Information at discrete pts can be misleading

$\sqrt{5}$ might actually be a local maximum $\Rightarrow$



Many people got this wrong

#6 b)



(note this just shows the direction of $h(x)$, we don't know that $h(x)$ is linear)

Since we know $h(x)$ is increasing up 'til $x=-4$, and decreasing from $x=-4$ to $x=-3$, then $h(x)$ has both a local and global maximum at $x=-4$ (Note, it doesn't matter that $h'(x)$ is undefined at $x=-4$, we know that $h(x)$ is continuous throughout $[-8,-3]$) Likewise $h(x)$ must have an absolute minimum at either $x=-8$ or $x=-3$ (or at both endpoints possibly). There are no local minimums.

#I Let x = height, y = depth. (And width = 2.)

Area $= 2(2\cdot x) + 2(2\cdot y) + 2(x\cdot y)$

$= 4x + 4y + 2xy$

Need an auxiliary eq'n to get rid of one variable.

Vol $= 10 = 2\cdot x\cdot y \Rightarrow y = \frac{10}{2x} = \frac{5}{x}$

Plug $y = 5/x$ into area eq'n.

Area $= 4x + 4(5/x) + 2x(5/x)$

$A(x) = 4x + \frac{20}{x} + 10$

$A'(x) = 4 - \frac{20}{x^2} = 0 \Rightarrow 4 = \frac{20}{x^2}$

$\Rightarrow x^2 = 5$

$\Rightarrow x = \sqrt{5}, \cancel{-\sqrt{5}}$ — x can't be negative since it's a length

And $A'(0)$ is undef'd but x can't be 0 because then volume $= 0$.

So, $x = \sqrt{5}$ is only critical pt. Check that it's a min.

A: $\searrow$ $\nearrow$

A': $-$ $\sqrt{5}$ $+$

OR $A''(x) = \frac{40}{x^3}$, so $A''(\sqrt{5}) > 0$

These show that $x = \sqrt{5}$ is a local min. But since $x = \sqrt{5}$ is the only crit. pt, it must also be the global min.

Dimensions: $y = \frac{5}{x} = \frac{5}{\sqrt{5}} = \sqrt{5}$. So, dimensions are $\boxed{\sqrt{5} \times \sqrt{5} \times 2}$.

- Many people forgot to prove $x = \sqrt{5}$ is a global (not just local) min
- Several people tried to minimize volume, but volume is fixed at 10; it's surface area we need to minimize.
- Several people set area $= 0$, but we want to set the derivative of area to zero in order to get the critical pts.
- Several people assumed the dimensions were $2\cdot x\cdot x$; this makes the assumption that one side is a square – a fact we were not given!