Dec. 6, '99

Solutions to 2nd Math Xa Midterm

① (a) false - inflection points can only occur where $g''(x)=0$, here $g''(e)=4\neq 0$

(b) true - we know that critical points include points where $g'=0$, here we're told $g'(e)=0$

(c) false - g would have to be concave down for this to be true at $x=e$, and $g''(e)=4$, (which is positive $\Rightarrow$ concave up)

(d) true - for the same reasons (c) is false. Since $g'(e)=0$ and $g''(e)=4$ then the second derivative test implies $x=e$ is a local minimum

(e) false - we already know $g(x)$ has a local minimum at $x=e$

(f) false - since the polynomial has a leading coefficient which is negative then $g(x)\to -\infty$ as $x\to\infty$, so $g(x)$ has no absolute minimum, only local minimums

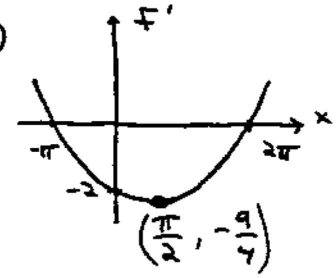
② try $\dfrac{(x-2)^4}{(x+1)\,x^2\,(x-1)} = Q(x)$

- has vertical asymptotes at $x=-1$, 0 and 1 because of the factors in the denominator,
- has just the one x-intercept at $x=2$ because that's the only root for the numerator
- is negative exactly where $(x+1)(x-1)$ is negative, since the other terms, x^2 and $(x-2)^4$ are always positive, and this is precisely when $-1<x<1$
- note $\lim\limits_{x\to\infty} Q(x) = \lim\limits_{x\to-\infty} Q(x) = 1$ because the ratio of the leading terms for numerator and denominator equals 1.

it's not necessary to do this problem for our fall 2000 MATH X course as of yet!

Solutions continued

③ (a)



(b) $f'(x) = k(x-(-\pi))(x-2\pi)$

and $f'(0) = -2 = k(0+\pi)(0-2\pi)$
$= k(-2\pi^2)$

so $k = \dfrac{-2}{-2\pi^2} = \dfrac{1}{\pi^2}$

so $f'(x) = \dfrac{1}{\pi^2}(x+\pi)(x-2\pi)$

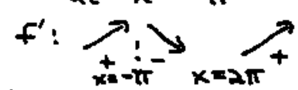
(c) x-coordinate of vertex is halfway between the two roots at $-\pi$ and 2π, at $\dfrac{2\pi+(-\pi)}{2} = \dfrac{\pi}{2}$

$f'\left(\frac{\pi}{2}\right) = \frac{1}{\pi^2}\left(\frac{\pi}{2}+\pi\right)\left(\frac{\pi}{2}-2\pi\right)$
$= \frac{1}{\pi^2}\left(\frac{3}{2}\pi\right)\left(-\frac{3}{2}\pi\right) = -\frac{9}{4}$

(d) $f'(x) = \frac{1}{\pi^2}(x+\pi)(x-2\pi)$
$= \frac{1}{\pi^2}x^2 - \frac{\pi}{\pi^2}x - \frac{2\pi^2}{\pi^2}$

so $f''(x) = \dfrac{2}{\pi^2}x - \dfrac{1}{\pi}$

(e) (i) use the first derivative test: f' goes from positive to negative at $x=-\pi$ $\Rightarrow$ local maximum



(ii) and likewise the only place that f' goes from negative to positive is at $x=2\pi \Rightarrow$ local minimum

(iii) since f' is a parabola, degree 2, then f must be a cubic polynomial, and cubics don't have abs. max or min ($\to$ range is $(-\infty,\infty)$)

④ (a) $W(100)$ is the amount of water in the reservoir after 100 seconds have passed

Solutions continued

(4) (b) $W'(200) = \frac{1}{2}$: When 200 seconds have passed by, water is coming into the reservoir at a rate of $1/2$ gallon per second

you did not have to do this one

(c) $W^{-1}(150)$ is the time at which there are 150 gallons of water in the reservoir

(5) (a) Since $(-\frac{1}{3}, 6)$ lies on the graph of f, then $f^{-1}(6) = -\frac{1}{3}$

(b) $f(x) = 5 \cdot e^{6x+2} + 1$, now rewrite as $x = 5 \cdot e^{6y+2} + 1$ and solve for y:

$x - 1 = 5 \cdot e^{6y+2}$, $\frac{x-1}{5} = e^{6y+2}$

$\ln\left(\frac{x-1}{5}\right) = \ln\left(e^{6y+2}\right) = 6y + 2$

so $y = \frac{1}{6}\left(\ln\left(\frac{x-1}{5}\right) - 2\right)$

$= f^{-1}(x)$

(c) so $f^{-1}(6) = \frac{1}{6}\left(\ln\left(\frac{6-1}{5}\right) - 2\right)$

$= \frac{1}{6}(\ln 1 - 2) = \frac{-2}{6} = \frac{-1}{3}$

(6) (a) $3 \log_7 (m^2 - 15) = 6$

$\log_7(m^2 - 15) = \frac{6}{3} = 2$

so $7^{(\log_7(m^2-15))} = 7^2 = 49$

$= m^2 - 15$

so $m^2 = 49 + 15 = 64$

and so $m = \pm 8$

(b) $7 \cdot 11^{m+1} = 5^m$

so $\log(7 \cdot 11^{m+1}) = \log(5^m) = m \cdot \log(5)$

$= \log(7) + \log(11^{m+1})$

$= \log(7) + (m+1)\log(11)$

$= \log(7) + m \cdot \log(11) + \log(11)$

Solutions continued

(6) (b) continued:

so $\log(7) + m \cdot \log(11) + \log(11) = m \cdot \log(5)$

so $m \cdot \log(11) - m \cdot \log(5) = -\log(11) - \log(7)$

and $m = \dfrac{-\log(11) - \log(7)}{\log(11) - \log(5)}$

$= \dfrac{\log(77)}{\log(5/11)}$ as well

(c) $4^{m^2} \cdot 4^{-m} + \log 2 = \log 4 + \log 5$

$4^{m^2 - m} = \log 4 + \log 5 - \log 2$

$= \log\left(\frac{4 \times 5}{2}\right) = \log 10 = 1$

$= 4^0$

so $4^{m^2 - m} = 4^0$, $m^2 - m = 0$

$m(m-1) = 0$

so $m = 0$ or $m = 1$

(7) (a) If $f(x) = x^5 - 2x^4 - 7$

Then $f'(x) = 5x^4 - 8x^3 = x^3(5x - 8)$

You know critical points are where either ① $f'(x) = 0$, ② $f'(x)$ is undefined or ③ endpoints of domain.

Thus the critical points for $f(x)$ on $[-1, 1]$ are $x = -1$, $x = +1$ (endpoints) and $x = 0$ (since $f'(0) = 0$)

[note $x = 8/5$, the other value where $f'(8/5) = 0$, is not in the domain of $f(x)$]

Now check sign of $f'(x)$:

f': [+   −]

shape of f: $-1 \nearrow 0 \searrow 1$

note: roots are at $x = 0$, $8/5$ for $f(x)$, so just need to check 1 point

for $x < 0$, $0 < x < 8/5$

$f'(-1) = 5 \cdot (-1)^4 - 8(-1)^3 > 0$

$f'(1) = 5 \cdot 1^4 - 8 \cdot 1^3 < 0$

Thus abs. max of $f(x)$ is $f(0) = -7$

(b) check endpoints for abs. min

$f(-1) = -10$ and $f(1) = -8$, so $f(-1) = -10$ is the abs. min for $f(x)$ on the interval $[-1, 1]$

(c) On $(-1, 1)$, the open interval, there are no endpoints $\Rightarrow$ no abs. minimums.

8. A, B, E, F, G

9a. $f(x) = 2000[1999^x - x^{1999}]$

$f'(x) = 2000[(\ln 1999)1999^x - 1999x^{1998}]$

b. $f(x) = \pi \ln\left(\frac{x^2}{\sqrt{3x}}\right)$

use log rules to simplify:

$f(x) = \pi(\ln x^2 - \ln\sqrt{3x})$

$f(x) = \pi(2\ln x - \frac{1}{2}\ln 3x)$

$f(x) = \pi(2\ln x - \frac{1}{2}\ln 3 - \frac{1}{2}\ln x)$

$f'(x) = \pi\left(\frac{2}{x} - 0 - \frac{1}{2x}\right)$

$= \frac{2\pi}{x} - \frac{\pi}{2x} = \frac{3\pi}{2x}$

common errors: NOT simplifying the log completely before differentiating

NOT distributing π

10 a. linear equation.

slope: $\frac{138-271}{7-0} = -19$

equation: $L(t) = -19t + 271$

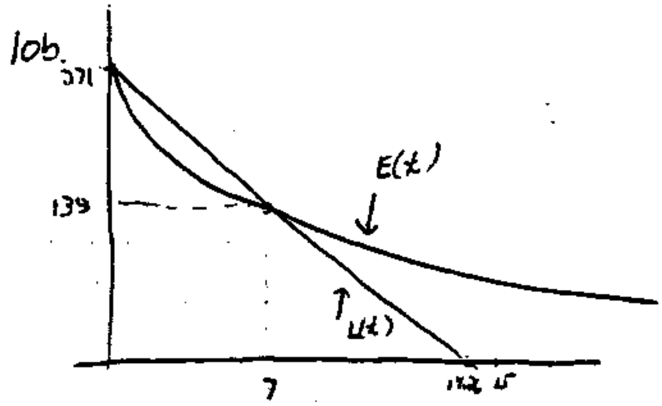
exponential equation: Find b:

$138 = 271 \cdot b^7$

$\frac{138}{271} = b^7$

$\left(\frac{138}{271}\right)^{1/7} = b$

$E(t) = 271 \cdot \left(\frac{138}{271}\right)^{t/7}$



common errors: not showing where the two graphs intersect. not providing enough detail (e.g. labelling intercepts)

C. linear model:

$0 = L(t) = -19t + 271$

$t = \frac{271}{19} = 14.26$ days

exponential model:

The substance never completely decays: the equation

$E(t) = 0 = 271\left(\frac{138}{271}\right)^{t/7}$

has no solutions.

Common error: trying to solve

$0 = 271\left(\frac{138}{271}\right)^{t/7}$ by taking ln (or log) of both sides.

This doesn't work because you can't take ln (or log) of 0.

10d. linear model:

$L'(t) = -19$ so $L'(3) = -19$

exponential model

$E'(t) = 271\left[\ln\left(\frac{138}{271}\right)^{1/7}\right]\cdot\left(\frac{138}{271}\right)^{t/7}$

$E'(3) = 271\cdot\left[\ln\left(\frac{138}{271}\right)^{1/7}\right]\cdot\left(\frac{138}{271}\right)^{3/7}$ (exact ans)

not necessary $\approx 271\cdot(-.0964)(.7488) = -19.56$

common errors: forgetting to take the derivative of E + L to find the rate of decay + plugging into the original equation;

forgetting to get an exact value for $E'(3)$

11a. Perimeter $= 2(10+2y)+2(8+2x)$

$= 36+4y+4x$

b. Matting:

Matting = area of entire frame − area of picture $= 200$

$200 = (2y+10)(2x+8) - 80$

solve for x:

$280 = (2y+10)(2x+8)$

$\frac{280}{2y+10} = 2x+8$

$\frac{140}{y+5} = 2x+8$

$\frac{70}{y+5} = x+4$

$\frac{70}{y+5} - 4 = x$

$x = \frac{50-4y}{y+5}$ now plug this in for x

$P(y) = 36+4y+4\left(\frac{50-4y}{y+5}\right)$

common error: using the wrong equation for matting

11c. To find the minimum perimeter

a) Find the critical pts. of $P(y)$

- take the derivative of P

the points where $P'(y)$ is undefined or zero are critical points

- the endpoints of the domain of P are also C.P. (although these endpoints do not give practical values in this problem)

b) Test each C.P. to see if it is a local min. Use the first derivative test by evaluating wether the sign of $P'(y)$ changes from neg. to positive at the critical pt.

$\overset{- - - -}{\longleftarrow}\underset{CP}{+}\overset{+ + + +}{\longrightarrow}$ (sign of P')

(For the C.P.'s where $P'(y)=0$ you may also use the 2nd. deriv. test)

CAUTION: there may be more than one local min. Which one minimizes the perimeter?

c) Plug all C.P.'s that give local minima back in to find the x values + then the perimeter. The C.P. which gives the smallest value for P is the right one to use.

Now plug the x + y values in to the dimensions to find the dimensions of the frame.